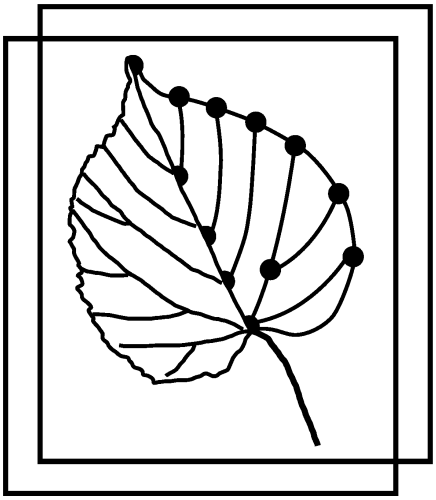
SPRING SCHOOL ON



COMBINATORICS 2007

Jan Hladký, Marek Krčál, Bernard Lidický (eds.)

Preface

Spring school on Combinatorics is a traditional meeting organized for members of the Combinatorial Seminar at Charles University for nearly 30 years. By now it is internationally well known and it is regularly visited by students, postdocs and teachers from our cooperating institutions in the DIMATIA network. In this year the Spring School was supported by Institute of Theoretical Computer Science (ITI).

Spring Schools are entirely organized and arranged by undegraduate students. The lectures' subjects are selected by supervisors from the Department of Applied Mathematics (KAM) and Institute for Theoretical Computer Science (ITI) of Charles University as well as from other participating institutions. In contrast, the lectures themselves are almost exclusively given by students, both graduate an undergraduate. This leads to a unique atmosphere of the meeting which helps the students in further studies and their scientific orientation.

This year the Spring School was organized for the third time in Vysoká Lípa, a picturesque village near romantic city of rocks Elbe Sandstones near the German border. Bizarre rock formations and narrow canyons of Kamenice river attracted a lot of interest of all participants, but have not distracted us from intensive scientific program.

We would like to thank Honza Hladký, Marek Krčál and Bernard Lidický as the main organizers who also completed this volume. We also thank Jiří Fiala, Pavel Valtr and other colleagues who took part both in the organization and in the Spring School itself. We hope to meet all this year's participants next year!

Dan Král', Jan Kratochvíl, Jaroslav Nešetřil,

Index of Talk Titles

Index of Speakers

Combinatorial Shifting

Dana Bartošová (with Marek Krčál)

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Presented paper by Gil Kalai

Intersection Patterns of Convex Sets

(*Israel Journal of Mathematics Vol. 48, Nos. 2-3, 1984, 161–174*)

The purpose of this talk is to show the Kalai's answer to the following question, raised by M. Katchalski and M. A. Perles:

> Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a family of n convex sets in R^d such that each $(d + r + 1)$-subfamily has empty intersection. What is the maximum possible number of $(d + 1)$-subfamilies (more generally, $(d+k)$-subfamilies) of $\mathcal{K}$ with nonempty intersection?

They conjectured that this maximum (for any k) is attained when $K_1 = \cdots = K_r = R^d$ and $K_{r+1}, \ldots, K_n$ are hyperplanes in general position. We show that this is indeed true.

Several applications will be mentioned in the talk of Marek Krčál.

More formally

Theorem 1. *Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a family of convex sets in R^d. For $0 \leq i < n$ denote by f_i the number of subsets S of $\{1, \ldots, n\}$ of cardinality $i + 1$ satisfying $\bigcap\{K_i \ : \ i \in S\} \neq \emptyset$.*

If $f_{d+r} = 0$ for some $r \geq 0$, then

$$f_{k-1} \leq \sum_{i=0}^{d} \binom{r}{k-i}\binom{n-r}{i} \quad \textit{for all } k > 0.$$

Equality holds, e.g., in the above mentioned situation.

Reformulation


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Definition 2. Lek $\mathcal{K} = \{K_\alpha : \alpha \in A\}$ be a family of sets. The *nerve* $N(\mathcal{K})$ of $\mathcal{K}$ is an abstract simplicial complex on A defined as

$$N(\mathcal{K}) = \{B \subseteq A : \bigcap_{\alpha \in B} K_\alpha \neq \emptyset\}.$$

Let C be a simplicial complex. C is called *d-representable*, if C is the nerve of a finite family of convex sets in R^d.

Definition 3. Let C be a simplicial complex and let S be a *free face* (i.e. a face of C of dimension $< d$, which is included in a unique maximal face of C). The operation of deleting S and every face that includes S is called an *elementary d-collapse*. A finite sequence of elementary d-collapses is called a *d-collapse*. C is *d-collapsible* if C d-collapses to the void complex.

Lemma 4. *Every elementary d-collapse can be replaced by a finite sequence of special elementary d-collapses of the following types:*

(A) Removal of a maximal face of dimension $< d$.

(B) Removal of a free face S of dimension $d-1$ and all the faces that include S.

We will use the following theorem of Wegner [1].

Theorem 5. *Every d-representable complex is d-collapsible.*

Let $n > r > 0, d, k$ be natural numbers. Let

$$\begin{aligned} p_k(n,d,r) &= \{S \in \{1,\ldots,n\}^{(k)} : |S \cap \{r+1,\ldots,n\}| \leq d\} \\ &= \sum_{i=0}^{d} \binom{r}{k-r}\binom{n-r}{i}. \end{aligned}$$

For a simplicial complex C, let

$$f_k(C) = |\{S \in C : \dim\ S = k\}|.$$

In view of this notation and Wegners's Theorem above, our theorem translates into the following:

Theorem 6. *Let C be a d-collapsible simplicial complex of dimension $\leq d+r-1$ $(r > 0)$ with n vertices. Then*

$$f_{k-1}(C) \leq p_k(n,d,r) \quad \textit{for all } k \geq 0.$$

The proof uses a correspondence between families of sets and subspaces of the exterior algebra.

References

[1] G. Wegner, d-collapsing and nerves of families of convex sets, *Arch. Math.* ***26*** *(1975), 317-327.*

Marek Krčál (with Dana Bartošová)

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Presented paper by Gil Kalai

Intersection Patterns of Convex Sets

(*Israel Journal of Mathematics Vol. 48, Nos. 2-3, 1984, 161–174*)

We will show some applications of the Katchalski-Perles conjecture proved in Dana's talk. Particularly, we will proof a sharp version of a theorem of Katchalski and Liu which says roughly the following: If $\mathcal{K}$ is a family of n convex sets in $\mathbf{R}^d$, and if *most* of its $\binom{n}{d+1}$ $(d+1)$-subfamilies have non-empty intersection, then $\mathcal{K}$ has an intersecting subfamily that contains *most* of members of $\mathcal{K}$. Other related results are also obtained.

Statements

Definition 1. For $0 \leq \rho \leq 1$ and $1 \leq d < k \leq n$, let $\alpha = \alpha(\rho, d, k, n)$ be the smallest real number with the following property:

(P) If $\mathcal{K}$ is a family of convex sets in $\mathbf{R}^d$, and if the number of intersecting k-subfamilies of $\mathcal{K}$ is $> \alpha\binom{n}{k}$ (i.e. $f_{k-1}(N(\mathcal{K})) > \alpha\binom{n}{k}$), then $\mathcal{K}$ has an intersecting subfamily of size $> \rho n$ (i.e. $1 + \dim N(\mathcal{K}) > \rho n$)

$\alpha(\rho, d, k, n) = \max\{\binom{n}{k}^{-1} f_{k-1}(N(\mathcal{K})) : \mathcal{K}$ is a family of n convex sets in $\mathbf{R}^d$ and $1 + \dim N(\mathcal{K}) \leq \rho n\}$

Definition 2. For $0 < \rho < 1$ and $d \geq 1$

$$\alpha(\rho, d, k) = \sup_{n \geq k} \alpha(\rho, d, k, n)$$

$$\alpha(\rho, d) = \sup_{k > d} \alpha(\rho, d, k)$$

Katchalski and Liu showed [2]:

$$\text{For any } \ d > 0 \text{ and } \rho < 1 : \quad \alpha(\rho, d) < 1.$$

They also proved that $\lim_{\rho \to 0} \alpha(\rho, d) = 0$. The exact value of $\alpha(\rho, d)$ was unknown for $d > 1$, but for $d = 1$ Abbot and Katchalski [1] proved that $\alpha(\rho, 1) = 1 - (1 - \rho)^2$.

Theorem 3. *For $0 < \rho < 1$, $0 < d < k$*

$$\alpha(\rho, d, k) = \sum_{i=0}^{d} \binom{k}{i} \rho^{k-i} (1 - \rho)^i = 1 - \sum_{i=d+1}^{n} \binom{k}{k} \rho^{k-i} (1 - \rho)^i.$$

In particular $\alpha(\rho, d) = \alpha(\rho, d, d + 1) = 1 - (1 - \rho)^{d+1}$.

Theorem 4. *Let $d \geq 1$ and $0 < c < 1$ be fixed. Then for each $0 < \varepsilon < 1$ there is a natural number $s = s(d, c, \varepsilon)$ such that every family of n convex sets in $\mathbf{R}^d$ with more than $\varepsilon \binom{n}{[cn]}$ intersecting subfamilies of size $[cn]$ has an intersecting subfamily of size $n - s$.*

An applications to extremal set theory

The condition of d-collapsibility in the main theorem (of Dana's talk) can be weakened.

Definition 5. A simplicial complex C is *weakly (d, r)-collapsible*, if C can be reduced to the void complex $\emptyset$ by a sequence of steps of the following two types:

(A') Removal of a maximal face of dimension $< d - 1$.

(B') Removal of a "weakly free" face S of dimension $\leq d - 1$ and all the faces that include S, where "S weakly free" means

$$\left| \bigcup \{T \in C : T \supset S\} \right| \leq d + r$$

(Note that if C is weakly (d,r)-collapsible, then $\dim C \leq d+r-1$.)

Theorem 6. *Let C be a weakly (d,r)-collapsible complex, with n vertices. Then*

$$f_{k-1}(C) \leq p_k(n,d,r) \quad \text{for all } k \geq 0.$$

This theorem can be rephrased as an extremal set theoretic result as follows:

Let $\mathcal{B}$ be a family of k-subsets of an n-set N. For $L \in N^{(d)}, d < k$ define

$$\deg_{\mathcal{B}} L = |\{C \in \mathcal{B} : L \subset C\}|.$$

Theorem 7. *Suppose $1 \leq d < k \leq d+r < n$. If $\mathcal{C}$ is a family of k-subsets of an n-set N and $|\mathcal{C}| > p_k(n,d,r)$, then there exists a nonempty subfamily $\mathcal{B}$ of $\mathcal{C}$ such that for all $L \in N^{(d)}$ either $\deg_{\mathcal{B}} L = 0$ or $|\cup\{C \in \mathcal{B} : L \subset C\}| > d+r$.*

References

[1] H. Abbot and M. Katchalski, A Turán type problem for interval graphs, *Discrete Math.* ***25*** *(1979), 85-88.*

[2] M. Katchalski and A. Liu, A problem of geometry in R^d, *Proc. Am. Math. Soc.* ***75*** *(1979), 284-288.*

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Presented paper by N. Alon and G. Kalai

A Simple Proof of the Upper Bound Theorem

(*Europ. J. Combinatorics (1985) 6, 211-214*)

Introduction

In this talk, we will prove the upper bound theorem (UBT), which says, that every d-dimensional polytope on n vertices has at most so many i-

dimensional faces as the cyclic d-polytope on n vertices. This proof applies for an arbitrary shellable triangulation of $(d-1)$-shpere.

On the Number of Elementary Collapses

Lemma 1. *Let $n > 1$, $N = \{1, 2, ..., n\}$ and $1 \leq s \leq m \leq n$. Let A_i and B_i (for $i = 1, ..., h$) be subsets of N such that $|A_i| \leq s$ and $|B_i| \geq m$ satisfying*

$$A_i \subseteq B_i, \ for\ 1 \leq i \leq h$$
$$A_i \not\subseteq B_j, \ for\ 1 \leq i < j \leq h$$

Then

$$h \leq \binom{n-m+s}{s}$$

Definition 2. Let C be a simplicial complex. A face $S \subseteq C$ is called *free* if it is contained in only one maximal face $M \subseteq C$. The operation of deleting S and all faces, that contain it is called an *elementary collapse*, or an *elementary (s,m)-collapse* where $|S| = s$ and $|M| = m$.

From the previous combinatorial lemma it immediately follows, that

Lemma 3. *if C is a simplicial complex and $s \leq m$ are nonnegative integers, then the number of all elementary (s',m')-collapses with $s' \leq s$ and $m' \geq m$ in any collapse process on C (which means a sequence of simplicial complexes, where the first complex is C and every other complex is an elementary collapse of the previous one) is at most $\binom{n-m+s}{s}$.*

The Upper Bound Theorem

We prove the UBT for simplicial polytopes since it is well known, that this is the worst case. In this case, the polytope forms a simplicial complex of its faces.

More precisely we prove the theorem for shellable shperes. Then, it is proven in [1], that every boundary complex of any simplicial convex polytope is shellable. This implies, that UBT holds for every convex polytope.

Definition 4. A simplicial complex C is called *shellable*, if all maximal faces $F \subseteq C$ have the same size d and can be ordered $F_1, F_2, ..., F_t$ so that for $1 \leq k \leq t-1$ holds

$$\overline{F}_k \cap \left(\bigcup_{i=k+1}^{t} \overline{F}_i \right) = \bigcup_{j=1}^{s_k} \overline{G}_j^k$$

Where $\overline{F}$ means the set of all subsets of F and G_j^k are distinct faces of C of size $d-1$.

d-collapsible Complexes

Definition 5. A simplicial complex C is called *d-collapsible*, if there exists a collapse process on C, where every elementary collapse is of type (d, m) for some $m \geq d$ and the final complex has no faces of size greater or equal d.

We prove a theorem, that

Theorem 6. *for $d \leq j \leq d+r-1$ let f_j denote the number of j-dimensional faces of a d-collapsible complex with n vertices. If $f_{d+r} = 0$ then*

$$f_j \leq \sum_{i=j+1-d}^{r} \binom{n-i}{d} \binom{i-1}{j-d}$$

References

[1] H. Bruggesser and P. Mani: Shellable decompositions of cells and spheres, *Math. Scand. 29*, 1971.

Counting and Sampling

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Presented paper by Mark Jerrum

Counting perfect matchings in a planar graph

(*Counting, sampling and integrating, Chapter 1.2*)

Introduction

We show a polynomial time algorithm for computing number of perfect matching in a planar graph. The algorithm is divided into two parts.

First introduce a so called Pfaffian orientation of a graph. From the Pfaffian orientation we are able to compute the number of perfect matchings using determinant of adjacency matrix.

Then we prove that every planar graph has a Pfaffian orientation. The proof is constructive and the Pfaffian orientation can be found in a polynomial time.

More formally

In graph $G(V, E)$ a *perfect matching* M is subset of edges such that every vertex has degree 1. Observe that a union of two matchings M and M' is a set of disjoint even cycles. Let C be an even cycle in G. We say that C is *oddly oriented* in orientation $\vec{G}$ if, when traversing C in either direction number of co-oriented edges is odd.

Definition 1. Graph G has a Pfaffian orientation $\vec{G}$ if every cycle, that can be created as union of two matchings is oddly orineted by $\vec{G}$.

For an orientation $\vec{G}$ we define *skew adjacency matrix* $A_S(\vec{G})$ as an adjacency matrix of G, where $A_{i,j}$ is changed from 1 to -1 if edge is orienter from j to i.

Theorem 2. *For any Pfaffian orientation $\vec{G}$ of G,*

$$\#\textit{perfect matchings in } G = \sqrt{\det A_S(\vec{G})}.$$

The problem with previous theorem is that we need to have a Pfaffian orientation of G. It is solvable in polynomial time for bipartite graphs. But in general it is not even known if it is in P or NP.

Fortunately for planar graphs the Pfaffian orientation is easy to find one.

Theorem 3. *Every planar graph has a Pfaffian orientation.*

It is sufficient to find an odd orientation of every face of G. Such an orientation is always Pfaffian.

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#P-completeness

(*Counting, sampling and integrating, Chapter 2*)

Classical complexity theory is mainly concerned with complexity of decision problems. The aim of my talk is to introduce the counting complexity class #P and the class of "most difficult" problems in #P, the so called #P-complete problems.

Definition 1. A counting problem $f : \Sigma^* \to \mathbb{N}$ is said to belong to the complexity class #P if there exist a polynomial-time predicate $\chi : \Sigma^* \times \Sigma^* \to \{0, 1\}$ and a polynomial p such that, for all $x \in \Sigma^*$,

$$f(x) = |\{\omega \in \Sigma^* : \chi(x, \omega) \wedge |\omega| \leq p(|x|)\}|.$$

It appears that NP-complete decision problems tend to give rise to #P-complete counting problems. What makes the theory of #P-completeness interesting is that there are #P-complete counting problems corresponding to easy decision problems. I show that following counting problem corresponding to polynomially solvable decision problem is #P-complete.

 # BIPARTITEPM

Instance: A bipartite graph G.

Output: The number of perfect matchings in G.

This problem is equivalent to computing the permanent of matrix.

0,1 PERM

Instance: A square 0,1-matrix $A = (a_{ij} : 0 \leq i, j < n)$.

Output: The permanent

$$perA = \sum_{\sigma \in S_n} \prod_{i=0}^{n-1} a_{i,\sigma(i)}$$

of A. Here, S_n denotes the symmetric group, that is, the sum is over all $n!$ permutations of $[n]$.

Theorem 2. *0,1 PERM (equivalently, #BIPARTITEPM) is #P-complete.*

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Presented paper by Mark Jerrum

Sampling and Counting

(*Counting, sampling and integrating, Chapter 3*)

Introduction

Even if some combinatorial counting problems are computationally difficult, that does not exclude the possibility of efficient approximate counting. In this chapter we demonstrate that approximate counting and almost uniform sampling are intimately related (if one can be done in polynomial time, then the other as well). The reduction from approximate counting to almost uniform sampling is illustrated by means of a concrete example: counting (not necessarily perfect) matchings in a graph.

Exercises

1. **Alternative** FPRAS **definition**
   A *fully polynomial randomized approximation scheme* for a counting problem $f : \Sigma^* \to \mathbb{N}$ is a randomized algorithm $N := \mathrm{FPRAS}(x, \varepsilon)$ (with $x \in \Sigma^*$, $\varepsilon > 0$) such that for all $x \in \Sigma^*$

$$\Pr[e^{-\varepsilon} f(x) \leq N \leq e^{\varepsilon} f(x)] \geq \frac{3}{4} \tag{1}$$

   holds, and the running time of $\mathrm{FPRAS}(x, \varepsilon)$ is bounded by a polynomial in $|x|$ and $1/\varepsilon$.
   Show that the definition can be equivalently written by replacing (1) with

$$\Pr[(1-\varepsilon) f(x) \leq N \leq (1+\varepsilon) f(x)] \geq \frac{3}{4}. \tag{2}$$

2. **Boosting the success probability of an** FPRAS
   Let $0 < \delta < 1$. How can the success probability of an FPRAS being boosted to

$$\Pr[e^{-\varepsilon} f(x) \leq N \leq e^{\varepsilon} f(x)] \geq 1 - \delta, \tag{3}$$

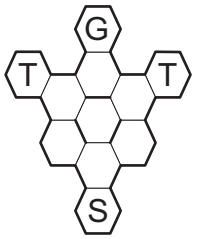
   with a runtime increase by a factor of $\mathcal{O}(\log(1/\delta))$? Consider taking the median and the sample mean of an appropriate number of repeated calls of the algorithm.

3. FPRAS $\leq$ FPAUS **for graph coloring**
   Let G be a graph of maximum degree Δ on n vertices, and let $\Omega(G)$ denote the set of all proper vertex q-colorings of G (for some $q > \Delta$). Show how an FPRAS for $|\Omega(G)|$ can be obtained from an FPAUS for $\Omega(G)$. There is no need to repeat all the calculations in the proof of Proposition 3.4, the key points are to derive relations akin to (3.4) and (3.5).

4. **Markov chains (conditional expectations)**
   A fair six-sided die is rolled over and over. What is the expected number of rolls until for the first time n sixes appear in a row ($n \geq 1$)?

5. **Markov chains (arrival probability, conditional probabilities)**
   A mouse is moving inside the labyrinth depicted below. In each time

step, it moves into one of the adjacent cells with equal probability (adjacent are cells that are not separated by a thick wall). What is the probability that the mouse reaches the cell G starting from S, without being caught by one of the traps T?



6. **Markov chains (stationary distribution)**
   A tank containing n gas molecules is separated into two halfs by a thin membrane. In each time step, a randomly chosen molecule crosses the barrier into the other half. Model this process as a Markov chain, and determine its stationary distribution (is it unique?). Is the Markov chain aperiodic?

7. **Markov chains (stationary distribution)**
   A random walk on an undirected, connected, non-bipartite graph $G = (V, E)$ is given by a sequence of moves of a particle between the vertices of G. If the particle is located at a vertex $v \in V$, it moves with probability $1/d(v)$ to a neighbor of v, where $d(v)$ denotes the degree of v. Show that the random walk is a Markov chain, and calculate its stationary distribution (is it unique?). Is the Markov chain aperiodic?

8. **Markov chains (stationary distribution)**
   Solve Exercise 3.10 from [1].

Solutions

1. From an FPRAS satisfying (1) we obtain an algorithm FPRAS$'$ as follows:

   FPRAS$'(x, \varepsilon')$ {
   $\varepsilon := \begin{cases} \varepsilon'/2 & \varepsilon' \leq 1 \\ 1/2 & \text{otherwise} \end{cases}$
   return FPRAS(x, ε)
   }

   For $\varepsilon' \leq 1$ it is

   $$(1-\varepsilon')f(x) \leq e^{-\varepsilon'/2} f(x) = e^{-\varepsilon} f(x) \leq N \leq e^{\varepsilon} f(x) = e^{\varepsilon'/2} f(x) \leq (1+\varepsilon')f(x), \quad (4)$$

   and for $\varepsilon' > 1$

   $$(1-\varepsilon')f(x) \leq e^{-1/2} f(x) = e^{-\varepsilon} f(x) \leq N \leq e^{\varepsilon} f(x) = e^{1/2} f(x) \leq (1+\varepsilon')f(x). \quad (5)$$

   Hence, $\Pr[e^{-\varepsilon} f(x) \leq N \leq e^{\varepsilon} f(x)] \geq \frac{3}{4}$ implies $\Pr[(1-\varepsilon')f(x) \leq N \leq (1+\varepsilon')f(x)] \geq \frac{3}{4}$.

   The running time of FPRAS$'(x, \varepsilon')$ is dominated by the running time of FPRAS(x, ε), which is polynomial in $|x|$ and $1/\varepsilon = 2/\varepsilon'$, therefore the running time of FPRAS$'$ is polynomial in $|x|$ and $1/\varepsilon'$ as well.

   To see the other direction, from an FPRAS satisfying (2) we obtain an algorithm FPRAS$'$ using the same pseudocode as above. Then for $\varepsilon' \leq 1$ it is

   $$e^{-\varepsilon'} f(x) \leq (1-\varepsilon'/2)f(x) = (1-\varepsilon)f(x) \leq N \leq (1+\varepsilon)f(x) = (1+\varepsilon'/2)f(x) \leq e^{\varepsilon'} f(x), \quad (6)$$

   and for $\varepsilon' > 1$

   $$e^{-\varepsilon'} f(x) \leq 1/2 f(x) = (1-\varepsilon)f(x) \leq N \leq (1+\varepsilon)f(x) = 3/2 f(x) \leq e^{\varepsilon'} f(x). \quad (7)$$

   Hence, $\Pr[(1-\varepsilon)f(x) \leq N \leq (1+\varepsilon)f(x)] \geq \frac{3}{4}$ implies $\Pr[e^{-\varepsilon'} f(x) \leq N \leq e^{\varepsilon'} f(x)] \geq \frac{3}{4}$.

   The argument for the bounded running time is identical.

2. Call the FPRAS $k := 2\lceil \log(1/\delta) \rceil + 1 = \mathcal{O}(\log(1/\delta))$ times and denote with M the median of the outputs $N_1, N_2, \ldots, N_k$. Then

$$\Pr[e^{-\varepsilon} f(x) \leq M \leq e^{\varepsilon} f(x)] = 1 - \Pr[M < e^{-\varepsilon} f(x) \vee M > e^{\varepsilon} f(x)] \tag{8}$$

$$\geq 1 - \Pr[M < e^{-\varepsilon} f(x)] \tag{9}$$

$$\geq 1 - \prod_{i=1}^{\frac{k+1}{2}} \Pr[N_i < e^{-\varepsilon} f(x)] \tag{10}$$

$$\geq 1 - (1/4)^{\frac{k+1}{2}} \tag{11}$$

$$\geq 1 - (1/e)^{\log(1/\delta)} \tag{12}$$

$$= 1 - \delta. \tag{13}$$

Note, that the sample mean can not be used instead, since already one of the outputs $N_1, N_2, \ldots, N_k$ falling outside the interval $[e^{-\varepsilon} f(x), e^{\varepsilon} f(x)]$ could cause the sample mean to be outside this interval.

3. For some enumeration $e_1, e_2, \ldots, e_m$ of the edges of G consider the graphs $G_i := (V(G), \{e_1, e_2, \ldots, e_i\})$ for $i = 0, 1, \ldots, m$. Then it is

$$|\Omega(G)| = q^n \frac{|\Omega(G_1)|}{|\Omega(G_0)|} \frac{|\Omega(G_2)|}{|\Omega(G_1)|} \cdots \frac{|\Omega(G_m)|}{|\Omega(G_{m-1})|} \tag{14}$$

(note, that $|\Omega(G_0)| = q^n$ holds). Define the ratios

$$\varrho_i := \frac{|\Omega(G_i)|}{|\Omega(G_{i-1})|} \tag{15}$$

for all $i = 1, 2, \ldots, m$.

Adding an edge to a graph does not increase the number of proper vertex colorings and therefore $|\Omega(G_i)| \leq |\Omega(G_{i-1})|$.

The colorings in $\Omega(G_{i-1}) \setminus \Omega(G_i)$ (they color the end vertices of e_i with the same color) can be mapped injectively to $\Omega(G_i)$ as follows: fix some arbitrary enumeration of the vertices of G, change the color of the lower index end vertex of e_i to a color that is not yet used in its neighborhood (a free color is available because of the assumption $q > \Delta$). Hence, $2|\Omega(G_i)| \geq |\Omega(G_{i-1})|$.

Together we get

$$\frac{1}{2} \leq \varrho_i \leq 1. \tag{16}$$

A similar calculation like in the proof of Proposition 3.4 with

$$N := q^n \bar{Z}_1 \bar{Z}_2 \cdots \bar{Z}_m \tag{17}$$

shows that

$$e^{-\varepsilon} \varrho_1 \varrho_2 \cdots \varrho_m \leq \bar{Z}_1 \bar{Z}_2 \cdots \bar{Z}_m \leq e^{\varepsilon} \varrho_1 \varrho_2 \cdots \varrho_m \tag{18}$$

holds with probability at least $\frac{3}{4}$, which implies

$$e^{-\varepsilon} \underbrace{q^n \varrho_1 \varrho_2 \cdots \varrho_m}_{=|\Omega(G)|} \leq N \leq e^{\varepsilon} \underbrace{q^n \varrho_1 \varrho_2 \cdots \varrho_m}_{=|\Omega(G)|} \tag{19}$$

with probability at least $\frac{3}{4}$.

4. We model this process as a Markov chain with states $S := \{0, 1, \ldots, n\}$ (the maximum number i of sixes that appeared in a row during the last i rolls). Let T_i denote the number of steps it takes to reach state n starting at i ($i = 0, 1, \ldots, n$), and let $t_i := \mathbb{E}[T_i]$. From conditional expectation laws we get the recursion

$$t_n = 0 \tag{20}$$

$$t_i = \frac{1}{6} t_{i+1} + \frac{5}{6} t_0 + 1 \qquad (i = 0, 1, \ldots, n-1). \tag{21}$$

Solving it gives $t_i = \frac{6}{5}(6^n - 6^i)$, in particular $t_0 = \frac{6}{5}(6^n - 1)$.

5. We denote the probability of reaching G starting from cell i with p_i. Using symmetry relations and conditional probability laws we obtain the following system of linear equations

$$p_S = p_1 \tag{22}$$

$$p_1 = 1/4(p_S + 2p_2 + p_3) \tag{23}$$

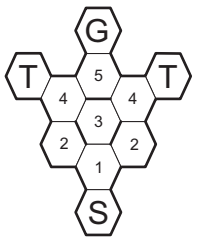
$$p_2 = 1/2(p_1 + p_4) \tag{24}$$

$$p_3 = 1/3(p_1 + 2p_4) \tag{25}$$

$$p_4 = 1/4(p_2 + p_3 + p_5 + 0) \tag{26}$$

$$p_5 = 1/3(2p_4 + 1). \tag{27}$$

Solving it gives $(p_S, p_1, p_2, p_3, p_4, p_5) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.



6. We let $S := \{0, 1, \ldots, n\}$ be the state space of the Markov chain (the number of molecules in one half of the tank). Then the transition probabilities are

$$\Pr[X_{t+1} = k \mid X_t = k-1] = \frac{n-k+1}{n} \qquad \text{for all } k = 1, 2, \ldots, n \tag{28}$$

$$\Pr[X_{t+1} = k \mid X_t = k+1] = \frac{k+1}{n} \qquad \text{for all } k = 0, 1, \ldots, n-1. \tag{29}$$

The Markov chain is finite and irreducible, it therefore has a unique stationary distribution π. We conjecture a binomial distribution

$$\pi_k = \binom{n}{k} 2^{-n} \qquad (k = 0, 1, \ldots, n). \tag{30}$$

And indeed,

$$(\pi P)_k = \frac{n-k+1}{n}\pi_{k-1} + \frac{k+1}{n}\pi_{k+1} \tag{31}$$

$$= \left(\frac{n-k+1}{n} \underbrace{\binom{n}{k-1}}_{\frac{k}{n-k+1}\binom{n}{k}} + \frac{k+1}{n} \underbrace{\binom{n}{k+1}}_{\frac{n-k}{k+1}\binom{n}{k}} \right) 2^{-n} \tag{32}$$

$$= \binom{n}{k} 2^{-n} \tag{33}$$

$$= \pi_k. \tag{34}$$

In the state diagram of the Markov chain all cycles have even length. Therefore the states have period length 2 (the Markov chain is not aperiodic).

7. Taking the vertex set of the graph as the state space of the Markov chain, it is clear that the particle position at every point in time depends only on the position in the previous time step. The Markov chain is finite (since the vertex set of the graph is finite) and irreducible (every vertex can be reached from every other vertex since the graph is connected), it therefore has a unique stationary distribution π. We conjecture

$$\pi_v = \frac{d(v)}{2|E|} \qquad (v \in V). \tag{35}$$

And indeed,

$$\begin{aligned}
(\pi P)_v &= \sum_{u \in \Gamma(v)} \frac{1}{d(u)} \pi_u && (36)\\
&= \sum_{u \in \Gamma(v)} \frac{1}{d(u)} \frac{d(u)}{2|E|} && (37)\\
&= \frac{1}{2|E|} \underbrace{\sum_{u \in \Gamma(v)}}_{=d(v)} && (38)\\
&= \pi_v, && (39)
\end{aligned}$$

where $\Gamma(v)$ denotes the set of vertices adjacent to v.

The shortest edge sequence from a vertex back to itself has length 2 (visit a neighboring vertex and go back). Since G is non-bipartite, there exists a cycle C of odd length. From every vertex $v \in V$, there is a path from v to a vertex $u \in C$ (G is connected). The edge sequence $v \to u \to C \to u \to v$ has odd length l. Therefore $\gcd(2, l) = 1$, every state has period length 1, the Markov chain is aperiodic.

8. Two matchings $M, M' \in \mathcal{M}(G)$ with $P(M, M') \neq 0$ differ in exactly one edge $e \in E(G)$ ($e \in M, e \notin M'$). Therefore $P(M, M') = \frac{1}{2}\frac{1}{|E(G)|} = P(M', M)$. With Lemma 3.7 this condition implies that the uniform distribution is the stationary distribution of this Markov chain.

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Presented paper by Mark Jerrum

Couplings and Colorings

(*Counting, sampling and integrating, Chapter 4.1, 4.2*)

Mixing time

We have seen how we can select random elements from some set by Markov chains. We start with some element, we run an ergodic Markov chain for sufficiently long time, and the distribution of the selected element eventually converges to the stationary distribution.

However, in practice we cannot affort to make infinite number of Markov steps. We therefore want to estimate the number of steps t such that after $\tau(\varepsilon)$ steps we are as close to the stationary distribution as any ε given in advance. Such a $\tau(\varepsilon)$ is called a *mixing time* of the Markov chain.

If the mixing time is polynomial in the size of input, we say that the Markov chain is *rapidly mixing*.

Couplings

Now we introduce *couplings*, which is a prominent tool for estimating the mixing time. We demonstrate their use on a simple Markov chain on colorings of a graph of small maximum degree Δ by $q \geq 2\Delta + 1$ colors. We show that this chain is rapidly mixing.

Let (Z_t) be a Markov chain with state space Ω. We interpret couplings as two Markov chains running in parallel universes in such a way that for an observer seeing only one chain, it seems to have the correct distribution. More formally, a coupling on (Z_t) is a Markov chain (X_t, Y_t) with state

space $\Omega \times \Omega$ satisfying

$$P(X_{t+1} = x' | X_t = x) = P(X_{t+1} = x' | X_t = x, Y_t = y)$$

and the analogous condition for Y.

However, the chains (X_t) and (Y_t) do not need to be independent. Instead we can choose the joint distribution so that the chains (X_t) and (Y_t) are encouraged to get closer to each other. We mostly use the method of coupling so that we start (X_t) in the state x, and we let (Y_t) start from the stationary distribution, and we use the following lemma:

Lemma 1. *[Coupling Lemma] Let (X_t, Y_t) be a coupling on a Markov chain (Z_t). Let a function $t : [0, 1] \to \mathbb{N}$ satisfy*

$$P(X_{t(\varepsilon)} \neq Y_{t(\varepsilon)} | X_0 = x, Y_0 = y) \leq \varepsilon, \qquad \text{for every } x, y \in \Omega,\ e > 0.$$

Then the mixing time $\tau(\varepsilon)$ is at most $t(\varepsilon)$.

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Path-coupling

(*Counting, sampling and integrating, Chapter 4.3*)

Coupling (as it was presented in chapter 4.2) is a very elegant and powerful method. What makes it hard to use, is the coalescence property together with the coupling property. These properties are often hard to define/verify. Therefore we establish the "adjacency-graph", whose vertices describe pairs of states of events we want to couple. Then we define coupling only for quadruples of states corresponding to vertices along an edge of this adjacency-graph. Main result of this chapter is a lemma that coupling can be propagated along paths in the adjacency-graph.

This method is an engineering approach to the coupling. We demonstrate this method by designing FPAUS that samples linear extensions of a given

POSET. We define edges in the adjacency graph between two linear extensions of a given POSET that differ by transposing two elements. We show following statements:

Lemma: For adjacent states X_0 and Y_0,

$$E[d(X_1, Y_1)|X_0, Y_0] \leq \rho d(X_0, Y_0),$$

where $\rho < 1$ is a constant depending on f. For a suitable choice for f, one has $\rho = 1 - \alpha$, where $\alpha = \frac{6}{n^3-n}$.

Lemma: Suppose a coupling (X_t, Y_t) has been defined as above on adjacent pairs of states and suppose that the couploing satisfies the contraction condition (on expected value from previous lemma) on adjacent pairs. Then the coupling can be extended to all pairs of states in such a way that the contraction condition holds unconditionally.

Proposition (Bubley and Dyer): The mixing time of the Markov chain on linear extensions (with transitions given by random choice of one element and transposing with its neighbor whenever this transposition forms a linear extension of the original POSET) is bounded by $\tau(\varepsilon) \leq \frac{n^3-n}{6}(2 \ln n + \ln \varepsilon^{-1})$.

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Presented paper by Mark Jerrum

Canonical paths

(*Counting, sampling and integrating, Chapter 5.1, 5.2*)

Introduction

The crucial point of using a Markov chain to create a FPAUS is to bound the mixing time of such Markov chain. One of the methods you can use is the Canonical paths method. The basic idea is to set up sort of a multicommodity flow problem. If we can find, for each two verticies, a path (with edges=transitions of the MC) between them, which isn't too long,

too improbable, and not many paths use a single edge, the chain mixes rapidly.

The intuition behind that is simple: if we had, for two verticies u, v, to use a long or improbable path to move between them, the mean time of getting to v starting in state u would be too big. The same applies if there are clouds of states in the MC with few edges between them (which corresponds to many paths sharing an edge).

More formally

Definition 1. A set of canonical paths Γ is

$$\Gamma = \left\{\gamma_{xy} | (x, y) \in \Omega^2\right\}$$

where γ_{xy} is a path between x and y.

Definition 2. The congestion of an edge (wrt a set of canonical paths Γ) is

$$\rho_{uv} = \frac{1}{\pi(u)P(u,v)} \sum_{\{xy | uv \in \gamma_{xy}\}} \pi(x)\pi(y)|\gamma_{xy}| \tag{40}$$

where u, v are two states with a transition between them[1].

Congestion of the whole MC is defined as maximum of ρ_{uv} over its transitions. It is a measure of how "bad" a particular set of canonical paths is.

Lemma 3. *For any random variable f on a probabilistic space Ω*

$$Var_\pi f = \frac{1}{2} \sum_{x,y \in \Omega} \pi(x)\pi(y)(f(x) - f(y))^2 \tag{41}$$

Definition 4. A Dirichlet form of a r.v. f is defined as

$$\mathcal{E}_P(f, f) = \frac{1}{2} \sum_{x,y \in \Omega} \pi(x)P(x,y)(f(x) - f(y))^2 \tag{42}$$

Comparing (2) and (3), one can take the Dirichlet form as the variance, where only neighboring states are considered.

[1] Throughout this abstract, $P(x, y)$ will be the transition probability between states x and y in the MC, and π its stationary distribution

Theorem 5. *For any r.v. f*

$$\mathcal{E}_P(f,f) \geq \frac{1}{\rho} Var_\pi f$$

Let's call Pf the r.v. $\sum_{y\in\Omega} P(x,y)f(y)$.

Lemma 6. *For any r.v. f and a lazy MC, ie. a MC whose transition matrix is defined as $\frac{1}{2}(P+I)$*

$$Var_\pi Pf \leq Var_\pi f - \frac{1}{2}\mathcal{E}_P(f,f) \tag{43}$$

Theorem 7. *The mixing time of the lazy Markov chain is bounded by*

$$\tau_x(\epsilon) \leq 2\rho \ \ln\frac{1}{\pi(x)\epsilon^2}$$

wehere ρ is the congestion of the original MC wrt any set of canonical paths.

The proof could be roughly described as follows: Combining Theorem 5 with Lemma 6, we obtain a bound on how fast Pf [2] converges to $\vec{1}$ [3]. This gives us a bound on the second eigenvalue and the eigenvalues of P and P^T are the same, so fast averaging (using P) means rapid mixing (with P^T). More on Dirichlet form and eigenvalues can be found in [2].

References

[1] MarkJerrum: Counting, sampling and integrating: algorithms and complexity 45-63

[2] David Aldous and James Fill: Reversible Markov Chains and Random Walks on Graphs http://www.stat.berkeley.edu/ aldous/book.html, chapter 3

[2] P is the transition patrix, st. $P^T\pi = \pi$

[3] Note π is the left eigenvector of P and $\vec{1}$ is the right eigenvector for the same eigenvelue

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Presented paper by Mark Jerrum

Canonical paths and sampling matchings of a graph

(*Counting, sampling and integrating, Chapter 5.3*)

In the talk we shall accomplish the original goal of the talk series. We show how to count approximately the number of matchings in a graph G in polynomial time. In order to do so, we consider the following Markov chain $\{X_t\}$ (on the state space $\Omega = \{M | M \text{ is a matching in } G\}$).

choose an edge $xy = e \in E(G)$ uniformly at random.
if $e \in X_t$ **then**
  $X'_{t+1} := X_t \setminus e$ *(Case "↓")*
else if both endvertices of e are unmatched in X_t **then**
  $X'_{t+1} := X_t \cup e$ *(Case "↑")*
else if the vertex x is unmatched and the vertex y is matched by an edge e' in X_t **then**
  $X'_{t+1} := X_t \setminus e' \cup e$ *(Case "↔")*
else
  $X'_{t+1} := X_t$
end if
with probability 1/2 set $X_{t+1} := X'_{t+1}$, and $X_{t+1} := X_t$ otherwise

It is not difficult to show that the Markov chain is irreducible and aperiodic and its stationary distribution is uniform distribution on the set Ω. We will show that the Markov chain mixes rapidly. Thus we have a FPAUS yielding a FPRAS (the reduction of the problem was shown in Torsten's lecture).

We will exhibit that the Markov chain mixes rapidly using the method of Canonical paths (the method of described in Jirka's talk).

Fix the ordering of the vertices of G. Between an initial matching I and a final matching F define a path $I = M_0, M_1, \ldots, M_k = F$ as follows. The symmetric difference $P = I \oplus F$ is a graph where every component is a path or an even cycle. Along the path, we modify the matching to look like in F step-by-step, starting the modifications in the component of P

containing the minimum element of the ordering. The modifications in each component of P are performed in an intuitive fashion; here we describe it only when the component C (we are currently working with) is a cycle. We take the minimum vertex v_0 in C. First we perform a $\downarrow$-operation on the edge which is adjacent to v_0 and contained in I. We continue with a sequence of $\leftrightarrow$-operations along the cycle and finish with one $\uparrow$-operation.

It is quite straightforward to bound the congestion of the defined system of canonical paths from above by a polynomial, thus yielding rapid mixing of the chain.

Let us note, that a slight modification of the chain $\{X_t\}$ gives a Markov chain for the problem of *weighted sampling* of the matchings. In the problem of weighted we want to sample every matching M with probability proportional to $\lambda^{|M|}$, where $\lambda > 0$ is a constant. The original problem thus corresponds to $\lambda = 1$. If λ was very large (exponentially in n), we get a uniform sampler for perfect matchings. Alas, the mixing time of the Markov chain is an increasing function of $\max\{\lambda, 1/\lambda\}$ and thus the mixing time the need not be polynomial anymore for the problem of sampling perfect matchings.


minimize
subj to
$x_1 \geq 0$
$x_2 \geq 0$
$x_1 + x_2 \geq 1$

Semidefinite Programming

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Presented paper by L.Lowasz

Semidefinite programs and combinatorial optimalization

(www.research.microsoft.com/ lovasz)

Introduction

In this paper we repeat basic results from linear algebra and define semidefinite program

Premilitaries

Here are basic results from linear algebra

Definition 1. Let A be square matrix. Eigenvector is vector x such $Ax = \lambda x$ we call associated λ eigenvalue.

Definition 2. The trace of matrix A is defined as $\sum_{i=1}^{n} A_{ii}$ Clearly λ is eigenvalue right when $det(A - \lambda I) = 0$ When A is symmetric $A = A^T$ Then all eigenvalues are real and eigenvectors form ortogonal basis. Then we write A as $U^T DU$ where U is orthogonal and D diagonal

Theorem 3. *(Interlacing eigenvalues) Let A be n*n symmetric matrix with eigenvalues $\lambda_1 \geq ... \geq \lambda_n$ Let B be an (n-k)*(n-k) symmetric minor with eigenvalues $\mu_1 \geq ...\mu_{n-k}$. Then $\lambda_i \leq \mu_i \leq \lambda_{i+k}$*

Definition 4. Matrix A is called positive semidefitite(definite) if all eigenvalues are nonnegative(positive).

Theorem 5. *For a real symmetric n*n matrix A, the following are equivalent (i) A is positive semidefinite (ii) the quadratic form $x^T Ax$ is nonnegative for every x*

Linear programming

Definition 6. linear program
maximize $c^T x$
subject to $Ax \leq b$

Definition 7. its dual
minimize $b^T y$
subject to $A^T y = c$
$y \geq 0$

Theorem 8. *(Duality Theorem) When one program has feasible solution so has other and optimums are equal*

Semidefinite programs

Definition 9. Semidefinite program is optimalization problem at form:
mininize $c^T x$
subject to $x_1 A_1 + x_2 A_2 + ... + x_n A_n - B$ semidefinite
because set of solutions is not closed we take infimum of objective function

Lemma 10. *Semidefinite farkas lemma The semidefinite program has no solution iff there exist semidefinite $Y \neq 0$ such that*

$$A_i \cdot Y = 0$$

$$B \cdot Y \geq 0$$

Definition 11. to primal program dual is
maximize $B \cdot Y$
subject to $A_i \cdot Y = c_i$
Y semidefinite

Theorem 12. *duality theorem if both programs have feasible solution $v_{primal} \geq v_{dual}$ If one has strictly feasible solution so has another, optimums are equal and attained.*

Semidefinite programming algorithm

We can solve semidefinite program by ellipsoid method in polynomial time.
We can also use interior point method for polynomial time.

References

[1] Lazlo Lovasz semidefinite programming, *www.research.microsoft.com/ lovasz*

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Presented paper by L. Lovász

Semidefinite programs and combinatorial optimisation II

(http://research.microsoft.com/users/lovasz/semidef.ps chapters 4 and 5)

Introduction

In this part of serie, we start by repeating the basics of semidefinite programming. After that we will examine some problems that can be solved by semidefinite programming.

Namely we will study unit distance representation of graph. Then we will learn how to relax some binary programs to vector programs, which are equivalent to semidefinite programs. We also show the connection between unit distance representation and $vartheta(G)$, which is parameter closely related to Shannon capacity.

More formally

Graph representations After repeating what is semidefinite programming, we will show following reduction of unit distance representation to semidefinite programming:

Lemma 1. *A graph G has a unit distance representation in ball of radius R (in some appropriately high dimension) if and only if there exists a positive semidefinite matrix A such that:*

$$\begin{aligned} A_{ii} &\leq R^2 \quad (i \in V) \\ A_{ii} - 2A_{ij} + A_{jj} &= 1 \quad (ij \in E). \end{aligned}$$

In other words, the smallest radius R is square root of optimum value of the

semidefinite program:

$$\begin{array}{lrcll} \textit{minimise} & w & & & \\ \textit{subject to} & A & \succeq & 0 & \\ & A_{ii} & \leq & w & (i \in V) \\ & A_{ii} - 2A_{ij} + A_{jj} & = & 1 & (ij \in E). \end{array}$$

We can slightly modify the program, such that all vectors will have same length R, by replacing inequality by equality. Then we have representation on the sphere. We can create orthogonal representation of $\overline{G}$ from this representation. Our unit distance representation lives in some $\mathcal{R}^n$. We add one dimension, and shift origin of the representation along the new e_{n+1}, so that length of all vectors increases to $R\sqrt{2}/2$. Then, vectors of vertices corresponding to edge in G are perpendicular. The new representation is optimal orthogonal representation of $\overline{G}$ and thus we can compute $\vartheta(\overline{G})$ this way.

Binary programming We also show how binary programs can be converted to strict quadratic programs and then relaxed to vector programs. This relaxation yields a bound and sometimes also good approximation of original problem. For binary programs (i.e., $x_i \in \{0, 1\}$) we use:$x_i^2 = x_i$.

Now we have quadratic program. We need strictly quadratic programs, so we multiply all linear terms by new variable u. We also add conditions $u^2 = 1$ and $ux_i \geq 0$. This ensures that u will be 1 for all feasible solutions. Having strict quadratic program, we can replace each variable with a vector with proper dimension, and express the term $x_i x_j$ as scalar product.

This is relaxation of quadratic program and it can be straightforward solved by semidefinite programming.

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Presented paper by László Lovász

Semidefinite programming and approximation algorithms

(http://research.microsoft.com/users/lovasz/notes.htm)

In the third talk based on Lovász's lecture notes on semidefinite optimization, we focus on usage of the technique in approximation algorithms. The basic idea here is the same as in the case of approximation algorithms based on linear programming. Instead of solving hard integer programs exactly, we construct an *semidefinite relaxation* of the original problem and find an optimal solution of it. Since this solution is typically not a feasible solution of the original problem, we used so-called *rounding techniques* to transform the optimal solution of the relaxation to a feasible solution of the original problem. However, we need the rounded solution be "close" to the original one, so that the value of the optimum is not decreased too much. Unlike linear programs, semidefinite programs can be used with constraints containing quadratic terms. Also, if a problem permits both linear and semidefinite relaxations, semidefinite relaxation may provide better approximation guarantee. In the talk, we present several examples where semidefinite relaxations are used and discuss the algorithmic aspects of the method.

Standalone Talks

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Presented paper by Jiří Fink

Matching graphs of Hypercubes

Abstract

Kreweras' conjecture [1] asserts that any perfect matching of the hypercube Q_d, $d \geq 2$, can be extended to a Hamiltonian cycle. We proved this conjecture in [2] but here we present an idea of a simplified proof. The *matching graph* $M(G)$ of a graph G is a graph whose vertex set consists of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings is a Hamiltonian cycle. We show that $M(Q_d)$ is bipartite. We proved that the matching graph of complete bipartite graph $K_{n,n}$, $n \geq 2$, is connected and bipartite if and only if n is even.

Introduction

A set of edges $P \subset E$ of a graph $G = (V, E)$ is a *matching* if every vertex of G is incident with at most one edge of P. If a vertex v of G is incident with an edge of P, we say that v is *covered* by P. A matching P is *perfect* if every vertex of G is covered by P.

The *d-dimensional hypercube* Q_d is a graph whose vertex set consists of all binary vectors of length d, with two vertices being adjacent whenever the corresponding vectors differ at exactly one coordinate. It is well known that Q_d is Hamiltonian for every $d \geq 2$. This statement can be traced back to 1872 [4]. Since then the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention. An interested reader can find more details about this topic in the survey of Savage [3], e.g. Dvořák [5] showed that any set of at most $2d - 3$ edges of Q_d ($d \geq 2$) that induces vertex-disjoint paths is contained in a Hamiltonian cycle. Dimitrov and Škrekovski [6] proved that for every perfect matching

P of Q_d ($d \geq 3$) there exists a Hamiltonian cycle that faults P, if and only if P is not a layer of Q_d.

The *matching graph* $M(G)$ of a graph G is a graph whose vertex set consists of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings is a Hamiltonian cycle. There is a natural one-to-one correspondence between Hamiltonian cycles of G and edges of $M(G)$ and enumerating of all Hamiltonian cycles of hypercube is well known open problem. Douglas [7] presents upper and lower bounds on the number of Hamiltonian cycles of hypercubes. So we are interesting in structural properties of $M(Q_d)$.

Kreweras [1] conjectured the following:

Conjecture 1. *Every perfect matching in the d-dimensional hypercube with $d \geq 2$ extends to a Hamiltonian cycle.* Kreweras' conjecture can be restated in the following way: *There is no isolated vertex in $M(Q_d)$, $d \geq 2$.*

Kreweras' conjecture

We show an idea of a proof of Kreweras' conjecture. Observe that the following theorem simply implies Kreweras' conjecture. For a given graph G we denote the complete graph on the vertices of G by $K(G)$. If the graph G is bipartite and connected, then we denote the complete bipartite graph on the vertices of G by $B(G)$.

Theorem 2. *[2] For every perfect matching P of $K(Q_d)$, $d \geq 2$, there exists a perfect matching R of Q_d such that $P \cup R$ is a Hamiltonian cycle of $K(Q_d)$.*

Proof The proof proceeds by induction on d. One observe that the statement holds for $d = 2$.

Let u_1u_2 be an arbitrary edge of P of $K(Q_d)$. We can divide the d-dimensional hypercube Q_d into two $(d-1)$-dimensional sub-hypercubes Q^1 and Q^2 such that $u_i \in V(Q^i)$ for $i \in \{1, 2\}$. Let $K^i = K(Q^i)$ and $P^i = P \cap E(K^i)$ for $i \in \{1, 2\}$.

The set of edges P^1 is a matching of K^1 which is not perfect. We denote by S^1 an arbitrary matching of K^1 which extend P^1 to a perfect matching. There exists a perfect matching R^1 of Q^1 such that $(P^1 \cup S^1) \cup R^1$ is a Hamiltonian cycle of Q^1 by induction. Observe, that $P^1 \cup R^1$ is a set of vertex-disjoint paths between vertices uncovered by P^1 and every vertex of Q^1 belongs to one of those paths.

Now, we extend the matching P^2 of K^2 to a perfect one by edges

$$S^2 = \left\{ xy \in E(K^2) \;\middle|\; \begin{array}{c} \exists x', y' \in V(Q^1) \text{ such that } xx', yy' \in P \\ \text{and there exists a path from } x' \text{ to } y' \text{ of } P^1 \cup R^1 \end{array} \right\}.$$

If there exists a path from x' to y' of $P^1 \cup R^1$ and $xx', yy' \in P$ and $xy \in E(K^2)$, then x' and y' are not covered by P^1 and x' and y' are vertices of both ends of a path of $P^1 \cup R^1$. Thus, the set of edges S^2 is a matching of K^2. Moreover, the set of edges $P^2 \cup S^2$ is a perfect matching of K^2 because S^2 covers each vertices covered by P but not by P^2. Hence, there must exist a perfect matching R^2 of Q^2 by the induction such that $(P^2 \cup S^2) \cup R^2$ is a Hamiltonian cycle of K^2. Now, it is easy to observe that $R = R^1 \cup R^2$ is a perfect matching of Q_d and $P \cup R$ is a Hamiltonian cycle. □

Let us note that we can consider the complete bipartite graph $B(Q_d)$ instead of complete graph $K(Q_d)$ in the last theorem. There would be only two small differences in the proof: We denote by S^1 an arbitrary matching of $B(Q^1)$ which extend P^1 to a perfect matching and we observe that S^2 is a matching of $B(Q^2)$.

Bipartite of $M(K_{n,n})$ and $M(Q_d)$

Let $G = (V, E)$ be a graph and $F \subseteq E$. We denote by $c(F)$ the number of components of a graph (V, F).

Theorem 3. *The matching graph $M(K_{n,n})$ is bipartite if n is even.*

Let us denote an arbitrary perfect matching of $K_{n,n}$ by Z. The color of perfect matching P of $K_{n,n}$ is $f(P) = c(P \cup Z) \pmod 2$. It easily follows from following lemma that f is proper vertex-coloring of $M(K_{n,n})$ by two colors.

Lemma 4. *For every perfect matchings P and R of $K_{n,n}$ it holds that*

$$f(P) + f(R) \equiv c(P \cup R) + n \pmod 2.$$

It is a natural question whether $M(K_{n,n})$ is bipartite also for n odd. The answer is negative for $n > 1$. We denote vertices of $K_{n,n}$ by $b_0, \ldots, b_{n-1}$ and $w_0, \ldots, w_{n-1}$.

Proposition 5. *Let $n \geq 2$ be a natural number and p be the smallest prime which divides n. Then $M(K_{n,n})$ contains a complete graph on p vertices as a subgraph.*

We observe that a union of every pair of the following perfect matchings gives us a Hamiltonian cycle:

$$P_i = \{b_j w_{j+i \bmod n} \mid 0 \leq j < n\}, \ 0 \leq i < p.$$

Theorem 6. *The graph $M(K_{n,n})$, $n \geq 2$, is bipartite, if and only if n is even.*

From the last theorem follows that $M(Q_d)$ is bipartite because it is a subgraph of $M(K_{2^{d-1},2^{d-1}})$.

Corollary 7. *The graph $M(Q_d)$, $d \geq 2$, is bipartite.*

Connectivity of $M(K_{n,n})$

Kreweras [1] considered a graph M_d which is obtained from $M(Q_d)$ by contracting every pair of vertices of $M(Q_d)$ whose corresponding perfect matchings are isomorphic. He conjectured the following:

Conjecture 8. *The graph M_d is connected.*

We again study the connectivity of $M(K_{n,n})$ first. The answer is simply negative for n odd. Let us consider the same coloring of vertices of $M(K_{n,n})$ as Theorem . From the Lemma follows that if P and R are perfect matchings of $K_{n,n}$ such that PR is an edge of $M(K_{n,n})$, then P and R have the same color.

Proposition 9. *The graph $M(K_{n,n})$ has at least two components if n is odd, $n \geq 3$.*

On the other hand we prove that there is no other component.

Lemma 10. *For every perfect matchings P and S of $K_{n,n}$ such that $c(P \cup S) \equiv n \pmod 2$ there exists a perfect matching R in $K_{n,n}$ such that $P \cup R$ and $R \cup S$ are Hamiltonian cycles of $K_{n,n}$, $n \geq 2$.*

This lemma gives an interesting property of $M(K_{n,n})$ that leads to determine even more than the number of components of $M(K_{n,n})$.

Theorem 11. *The distance between perfect matchings P and S in the graph $M(K_{n,n})$ is following:*

	$c(P \cup S)$	n *is even*	n *is odd*
	$= n$	*0*	*0*
	$= 1$	*1*	*1*
$1 < c(P \cup S) < n$	$\equiv n \pmod 2$	*2*	*2*
	$\equiv n + 1 \pmod 2$	*3*	∞

Corollary 12. *The graph $M(K_{n,n})$, $n \geq 2$, has one component for n even and two components for n odd.*

Open problem

Riste Škrekovski [8] asked whether the following stronger form of Kreweras' conjecture could be true:

Does every (not necessarily perfect) matching of Q_d, $d \geq 2$, extends to a Hamiltonian cycle of Q_d.

The statement can be shown to be true for $d = 2, 3, 4$. However, our approach does not seem to lead to proving this stronger statement.

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For the study of the properties of $M(Q_d)$, one might ask what additional requirements can we pose on the extending perfect matching R in Theorem . For example, can we find R that satisfies Theorem and contains only edges from a given list of layers of hypercube? A natural necessary condition says that the set D of allowed edges for R together with the prescribed matching P form a connected subgraph. The following result due to Gregor [9] shows that this condition is also sufficient in the case when D is formed by disjoint subcubes of (possibly different) nonzero dimensions.

For our convenience, let us consider $P \cup R$ where $P = R = \{\{0, 1\}\}$ to be a Hamiltonian cycle of $K(Q_1)$, so Theorem holds for $d \geq 1$.

Theorem 13.*[9] Let $A_1, \ldots, A_m \subseteq V(Q_d)$ ($d \geq 1$) be pairwise disjoint subcubes of nonzero dimension. Let $A = \bigcup_{i \in [m]} A_i$, $D = \bigcup_{i \in [m]} E(A_i)$ and let P be a perfect matching of $K(A)$. There exists $R \subseteq D$ such that $P \cup R$ is a Hamiltonian cycle of $K(A)$ if and only if $P \cup D$ is connected.*

*

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A removal lemma for groups

(http://www.arxiv.org/pdf/math.CO/0310476)

Abstract

The removal lemma for graphs is a well-known and important combinatorial tool with many applications. In the talk we will present an analogous formulation of the lemma for groups introduced by Ben Green in 2004. We will discuss the similarities with the original lemma and some of it's applications.

Introduction

Definition 1.[Triangles in an abelian group] Let G be an abelian group, $A \subseteq G$, T will be the triangles in A: $T = \{(a, b, c) \in A^3 | a + b + c = 0\}$.

From the Szemerédi Regularity Lemma one can prove the removal lemma for graphs:

Lemma 2.*[Removal Lemma (Ruzsa-Szemerédi)] Let $G = (V, E)$ a graph $|V| = n$. If G has $o(n^3)$ triangles (K_3) then there is $E' \subset E$ such that $|E'| = o(n^2)$ and makes $G' = (V, E \setminus E')$ triangle-free.*

However a similar lemma for abelian groups can be introduced.

Theorem 3. *Let G be an abelian group, $|G| = n$, let $A \subseteq G$, $|A|$. If the number of triangles in (A) is $o(n^2)$ then exist $A' \subseteq A$, with $|A'| = o(n)$ such that $A \setminus A'$ is triangle-free.*

As for graphs a regularity lemma for groups is proved previously (see [1]).

Proof [Sketch proof in the $(\mathbf{Z}/2\mathbf{Z})^N$ case] Let A_H^{+x}, $x \in G$, be the indicator function of the set $A \cap (H + x)$ and $\widehat{A_H^{+x}}(\gamma)$ the Fourier tranform of A_H^{+x}. As for graphs a regularity notion is introduced; x is ϵ-regular for the set A and the subgroup H if $\sup_{\gamma \neq 0, \gamma \in H^*} |\widehat{A_H^{+x}}(\gamma)| \leq \epsilon |H|$. If the number of $x \in G$ which fail to be ϵ-regular (respect to the subgroup H and the set A) is no more than ϵn then we say that H is ϵ-regular for A. The indicator function is also introduced: $ind(A; H) = 1/n \sum_{x \in G} \left(|A_H^{+x}|/|H|\right)^2$. This function is positive and bounded from above. Let $H_0 = G$. Whenever H_i is no regular for A, it is possible to find $H_{i+1} \leq H_i$ subgroup, with bounded index (depends on H_i index) such that $ind(A; H_{i+1}) > ind(A; H_i) + f(\epsilon)$ for a fixed $f > 0$. So this process should end finding an appropiate H. □

In the $(\mathbf{Z}/2\mathbf{Z})^n$ case the ϵ^2-regularity for groups can be interpreted as the ϵ-regularity for graphs in a certain Cayley graph over $(\mathbf{Z}/2\mathbf{Z})^N$ with the partition induced by the subgroup H and its cosets.

In fact a more general theorem can be proved.

Theorem 4. *Let $k \geq 3$ be a fixed integer, and suppose that $A_1, \ldots, A_k$ are subsets of G such that there are $o(n^{k-1})$ solutions to the equation $a_1 + \ldots + a_k = 0$ with $a_i \in A_i$ for all i. Then we may remove $o(n)$ elements from each A_i so as to leave sets A_i' such that there are no solutions to $a_1' + \ldots + a_k' = 0$ with $a_i' \in A_i'$ for all i.*

Corollary 5.*[Application to sum-free sets] Suppose that $A \subseteq 1, \ldots, n$ is a set containing $o(n^2)$ triples with $x + y = z$, $x, y, z \in A$. There is a subset $A' \subset A$ with $|A'| = o(n)$ such that $B = A \setminus A'$ is sum-free.*

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Probabilistic model checking – pPDA

We give a survey of some basic model checking problems and their solutions for stochastic systems generated by probabilistic pushdown automata.

Probabilistic pushdown automata

A *pushdown automaton (PDA)* consists of a finite-state control unit and one unbounded stack. The *configuration* of a PDA is given by its current state and the content of the stack, which is a finite word over a finite *stack alphabet*. We write it $p\alpha$ where p is the state and α the stack content (top=left). Assuming the stack is not empty, we may write $\alpha = X\beta$, where X is the top-most symbol on the stack, and we denote pX the *head* of $pX\beta$. The computation of a PDA is driven by a finite set of *transition rules*. Every

rule is of the form $pX \to q\beta$, where p, q are states, X is a stack sybmbol and β is a string of at most 2 stack symbols. These rules describe the (infinite) transition relation – for any configuration $pX\alpha$ and for any rule $pX \to q\beta$ there is a transition $pX\alpha \to q\beta\alpha$. We require that for every head there is at least one transition rule. Denoting ε the empty stack, we set by definition that for every state p the only possible transition from $p\varepsilon$ is to $p\varepsilon$.

Restricting the set of control states to be a singleton, we can completely omit it and we get the class of *stateless PDA*. It is denoted BPA, since it describes the same class of processes as the Basic process algebra.

Note that the PDA is nondeterministic, i.e. there can be several different rules applicable to the current configuration. Solving this nondeterminism by giving a probability distribution over the applicable rules we get a *probabilistic PDA (pPDA)*. Similarly we define *pBPA*. Writing $p\alpha \xrightarrow{x} q\beta$ means that the rule inducing the transition $p\alpha \to q\beta$ was given probability x.

Probabilistic PDA generates a special infinite-state stochastic process called *discrete homogenous Markov chain*. The class od processes generated by pPDA is also known as the class of Recursive Markov chains. A sequence of configurations $w = p_1\alpha_1, p_2\alpha_2, \ldots$ where $p_i\alpha_i \to p_{i+1}\alpha_{i+1}$ is a legal transition for every i is called a *path*. By w_i we denote the path $p_i\alpha_i, p_{i+1}\alpha_{i+1}, \ldots$. Infinite paths are called *runs*. Let w be a finite path. The set of all runs beginning with w is denoted $Run(w)$.

A *σ-algebra* over a set M generated by some $\mathcal{A} \subseteq 2^M$ is the least set $\mathcal{F} \supseteq \mathcal{A} \cup \{\emptyset\}$ closed on complement and countable unions. A map $\mathcal{P} : \mathcal{F} \to [0, 1]$ is called a *probability measure* if $\mathcal{P}(\emptyset) = 0$, $\mathcal{P}(M) = 1$ and for any countable sequence $A_1, A_2, \ldots$ of pairwise disjoint elements of $\mathcal{F}$: $\mathcal{P}(\bigcup_i A_i) = \sum_i \mathcal{P}(A_i)$. We call $\mathcal{F}$ the set of *events* and the pair $(\mathcal{F}, \mathcal{P})$ a *probabilistic space*.

Any configuration $p\alpha$ can be assigned a σ-algebra $\mathcal{F}$ over $Run(p\alpha)$ generated by all sets $Run(w)$ where w is a finite path beginning with $p\alpha$. Slightly abusing the notation let $w = p\alpha \xrightarrow{x_1} p_1\alpha_1 \xrightarrow{x_2} \ldots \xrightarrow{x_n} p_n\alpha_n$. There is a unique probability measure $\mathcal{P}$ on $\mathcal{F}$ satisfying $\mathcal{P}(Run(w)) = \prod_{i=1}^{n} x_i$. Thus, any configuration is assigned a probabilistic space of runs from it.

Objectives

An *objective* is a property of a Markov chain. Given a pPDA the goal is to decide, whether the pPDA satisfies the objective. We deal with two types of objectives, linear and branching ones.

A *(quantitative) linear objective* is given by a starting configuration $p\alpha$, some property of runs from $Run(p\alpha)$ and a rational q. Solving the objective means to decide, whether the event that a run satisfies the given property has probability $\leq (\geq, <, >)\ q$. If $q \in \{0, 1\}$ we talk about a *qualitative* linear objective.

Of course, we must carefully check that the set of all runs satisfying the property forms an event. Fortunately, this is the case of all of the following properties.

The simplest property is the *termination*. A run w terminates if it visits a configuration with an empty stack. More general is the reachability property: given a set R of configurations, we ask, whether the run visits a configuration from R. For algorithmic issues we require the set R to be regular. Next step are the properties specified by the LTL logic. Even more general is the ω-regular property. Consider the finite alphabet of all heads and all configurations with the empty stack. Given some ω-regular language over this alphabet, we can ask, whether the word induced by a given run belongs to it or not. According to how the language is given, we speak about nondeterministic Büchi ω-regular properties, determinisic Rabin ω-regular properties etc.

For pPDA, solving termination is up to a polynomial time reduction equivalent to solving reachability. We do not know, whether this holds for pBPA.

The following table shows the upper bounds in complexity of solving the mentioned objectives. Note, that nondeterministic Büchi ω-regular properties are **PSPACE**-hard already for systems generated by finite probabilistic automata instead of pPDA.

reachability	**PSPACE**
LTL	**PSPACE** in pPDA, **EXPSPACE** in formula
det. Rabin	**PSPACE**
nondet. Büchi	**PSPACE** in pPDA, **EXPSPACE** in Büchi aut.

For qualitative linear objectives, the upper bounds differ often for pBPA, as the next table shows:

reachability	**PSPACE**
termination pBPA	**PTIME**
reachability pBPA	**PTIME**
LTL	**PSPACE** in pPDA, **EXPTIME** in formula
LTL pBPA	**PTIME** in pBPA, **EXPTIME** in formula
det. Rabin	**PSPACE** in pPDA, **PTIME** in Rabin aut.
det. Rabin pBPA	**PTIME**
nondet. Büchi	**PSPACE** in pPDA, **EXPTIME** in Büchi aut.
nondet. Büchi pBPA	**PTIME** in pBPA, **EXPTIME** in Büchi aut.

Moreover solving LTL objectives as well as nondeterministic Büchi objectives for pBPA is **EXPTIME**-hard, thus we get **EXPTIME**-completeness.

Fixing a linear objective up to the initial configuration, we can define sets of configurations for which the given linear objective is satisfied. For qualitative ω-regular objectives this set is always effectively regular. This fails, however, already in the case of pBPA and quantitative termination objective.

Another kind of objectives are *branching* objectives. Such an objective is specified by a formula of some probabilistic branching time logic. The model checking problem is, given a starting configuration $p\alpha$, to determine, whether the formula is valid in it.

In a probabilistic branching time logic there are usually two kinds of formulae: state formulae (Φ in the abstract syntax equations below) and path formulae (Ψ). We start with the logic PCTL*. Its syntax follows:

$$\begin{aligned} \Phi &::= a \mid \Phi \vee \Phi \mid \neg\Phi \mid \mathcal{P}^{\sim p}\Psi \\ \Psi &::= \Phi \mid \Psi \vee \Psi \mid \neg\Psi \mid \mathcal{X}\Psi \mid \Psi\mathcal{U}\Psi \end{aligned}$$

Here a denotes any atomic proposition, $\sim \in \{=, \neq, >, <, \leq, \geq\}$ and p is a rational between 0 and 1.

The logic PCTL$^+$ is a fragment of PCTL* without nested operators $\mathcal{X}$ and $\mathcal{U}$ in path formulae. Another logic, PCTL, is a fragment of PCTL$^+$ where path formuale do not even contain boolean connectives.

The last logic we introduce is the logic PECTL*. Its syntax is given by:

$$\Phi ::= a \mid \Phi \vee \Phi \mid \neg\Phi \mid \mathcal{P}^{\sim p}\mathcal{B}(\{\Phi, \ldots, \Phi\})$$

where $\mathcal{B}$ is a Büchi automaton. Denoting F the set of formulae given as the argument of $\mathcal{B}$, we require $\mathcal{B}$ to work with the alphabet 2^F. Observe that there are no path formulae in PECTL*.

We also define the *qualitative* fragments qPCTL*, qPCTL$^+$, qPCTL, qPECTL* of the logics defined above by allowing only $p \in \{0, 1\}$ for $\mathcal{P}^{\sim p}\Psi$ formulae.

The semantics of a state formula is a subset of configurations, the semantics of a path formula is a subset of runs satisfying it. Atomic propositions are assumed to have regular valuations, boolean connectives mean the usual set operations. The operator $\mathcal{P}$ is a kind of stochastic quantification, the formula $\mathcal{P}^{\sim p}\Psi$ is true in a configuration $p\alpha$ iff the probability measure of runs from $Run(p\alpha)$ satisfying Ψ is $\sim p$. A run beginning with the configuration $p\alpha$ satisfies a state formula φ iff $p\alpha$ satisfies it. A run w satisfies $\varphi\mathcal{U}\psi$ iff there is i such that w_i satisfies ψ and for all $j, 0 < j < i$ w_j satisfies φ. A run w satisfies $\mathcal{X}\varphi$ iff w_2 satisfies φ. Every ω-regular automaton $\mathcal{B}$ from PECTL* is associated with a set F of state formulae. To every run $w = w(1), w(2), \ldots$ we assign the infinite word $\bar{w} = \bar{w}(1), \bar{w}(2), \ldots$ where $\bar{w}(i) \subseteq F$ are the formulae satisfied in $w(i)$. Then w satisfies $\mathcal{B}(F)$ iff $\bar{w}$ is accepted by $\mathcal{B}$.

Note that, although with simpler syntax, PECTL* semantically covers PCTL*. Indeed, there is an algorithm for translating any LTL formula into an equivalent Büchi automaton. The size of the automaton is exponential in the size of the formula. Now this algorithm can easily be modified to translate PCTL* formulae to PECTL* with an exponential blow-up.

The qualitative fragments defined above are essential, since already model checking PCTL for pPDA and PCTL$^+$ for pBPA is undecidable. However, restricting to the qualitative case renders it decidable even for qPECTL*.

There is an algorithm for model checking pPDA against qPECTL* which runs in time doubly exponential in the size of the formula and exponential in the size of the pPDA. For a fixed formula and a pBPA it runs even in polynomial time.

If only automata from a fixed finite set can be used in the formula, then the algorithm runs in only singly exponential time in the size of the formula. Especially for qPCTL we derive the **EXPTIME** upper bound, since $\mathcal{X}$ and $\mathcal{U}$ operators can be encoded into Büchi automata. Interestingly, there is also a matching lower bound even for model checking pBPA against qPCTL, so qPCTL model checking is **EXPTIME**-complete.

Translating qPCTL* into qPECTL* costs an exponential blow-up in the size of the formula, so we have a triple exponential upper bound on model

checking qPCTL*.

Resources and further work

All facts mentioned in this survey can be found, along with detailed proofs, in the PhD thesis of Tomáš Brázdil (see e.g. `http://www.fi.muni.cz/~xbrazdil`). The thesis contains also a deep analysis of more complex model checking questions regarding pPDA processes, like the expected gain and long run properties.

Interesting results and open problems arise when we extend pPDA to $1\frac{1}{2}$-player stochastic games. This model allows both stochastic as well as nondeterministic transitions. See e.g. the work of Kučera et al., Reachability in Recursive Markov Decision Processes, in proceedings of Concur 2006, LNCS 4137, 358-374.

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Presented paper by Andrew Thomasson and Peter Wagner

Complete graphs with no Rainbow Path

(`www.intersciens.wiley.com`)

Abstract

Motivated by questions in Ramsey theory, we consider colorings of the edges of the complete graph K_n that contain no rainbow path P_{t+1} of length t. If fewer than t colors are used then certainly there is no rainbow P_{t+1}.We show that, if at least t colors are used, then very few colorings are possible if $t \leq 5$ and these can be described precisely, whereas the situation for $t \geq 6$ is qualitatively different.

Introduction

We call the path P_{t+1}, of order $t+1$ and having t edges, a t-path. We consider colorings of the edges of the complete graph K_n that have no

rainbow t-path (where by a rainbow copy of the graph H, we mean one in which all edges have different colors).

The classical t-color Ramsey number of the graph G is the minimum n such that, whenever the edges of K_n are colored with at most t colors, there must be a monochromatic copy of G.

The t-local Ramsey number of G, introduced by Gyárfás, Lehel, Schelp, and Tuza [5], is the minimum n such that, whenever the edges of K_n are colored so that no more than t colors appear at any vertex, there must be a monochromatic copy of G. The t-local colorings may be viewed as those forbidding a rainbow $K_{1,t+1}$.

Gyárfás, Lehel, and Schelp in [4] investigated colorings in which rainbow paths, rather than stars, are forbidden: these are exactly the colorings we discuss in this note. More generally, Jamison, Jiang, and Ling [7] introduced the parameter $f(G, H)$, the minimum n such that every edge-colored K_n contains either a monochromatic copy of G or a rainbow copy of H. It is readily seen from the Canonical Ramsey theorem of Erdos and Rado [2] that $f(G, H)$ exists if and only if G is a star or H is a forest.

It was shown in [4] that, for $|G| \geq 5$, $f(G, P_4)$ is equal to the ordinary 2-color Ramsey number of G; that is, forbidding rainbow 3-paths does not allow any larger colorings than simply using only 2 colors. It was also shown that, for most G, $f(G, P_5)$ is equal to the ordinary 3-color Ramsey number of G. These results follow straightforwardly from Theorems 1 and 2 below because, even without regard to the graph G, for $t \leq 5$ it is very difficult to color K_n with t or more colors without creating a rainbow t-path. On the other hand, we show that it is much easier to avoid a rainbow t-path for $t \geq 6$.

We remark too that the function $f(K_{1,s}, K_t)$ has been investigated by Alon, Jiang, Miller, and Pritikin [1], who proved $f(K_{1,s}, K_t) = \Theta(st^3/\log t)$. For $t = 3$, we have the exact result $f(K_{1,s}, K_t) = \lceil 5s/2 \rceil - \delta$, where $\delta = 2$ if s is odd and $\delta = 3$ if s is even.

We use natural numbers as colors. Given an edge coloring of K_n, let E_j be the set of edges colored j and let V_j be the vertices at which an edge of color j appears.

To begin, observe that if K_n has no rainbow 2-path P_3 then the coloring is monochromatic. The situation with no rainbow 3-path is only slightly less restrictive.

Theorem 1.. *Let K_n, $n \geq 4$, be edge colored so that it contains no rainbow 3-path P_4. Then one of the following holds:*

a. at most two colors are used;

b. $n = 4$ and three colors are used, each color forming a 1-factor.

The colorings that avoid a rainbow 4-path are listed in the next theorem. There are more possibilities but they are still very restricted; amongst the colorings using more than three colors, the type that allows the most variation is (b); in this coloring, there is a special color, and each vertex is allowed incident edges of only one color apart from the special color. Only types (b) and (c) allow more than four colors, and types $(c) - (f)$ all contain a monochromatic K_{n-3}.

Theorem 2.. *Let $K_n, n \geq 5$, be edge colored so that it contains no rainbow 4-path P_5. Then, after renumbering the colors, one of the following must hold:*

a. at most three colors are used;

b. color 1 is dominant, meaning that the sets $V_j, j \geq 2$, are disjoint;

c. $K_n - a$ is monochromatic for some vertex a;

d. there are three vertices a, b, c such that $E_2 = \{ab\}, E_3 = \{ac\}$, E_4 contains bc plus perhaps some edges incident with a, and every other edge is in E_1;

e. there are four vertices a, b, c, d such that $\{ab\} \subset E_2 \subset \{ab, cd\}$, $E_3 = \{ac, bd\}$, $E_4 = \{ad, bc\}$ and every other edge is in E_1;

f. $n = 5, V(K_n) = \{a, b, c, d, e\}$, $E_1 = \{ad, ae, bc\}$, $E_2 = \{bd, be, ac\}$, $E_3 = \{cd, ce, ab\}$ and $E_4 = \{de\}$.

The colorings that avoid a rainbow 5-path P_6 are more varied, but they can still be described explicitly. A list of the possibilities, broken down into 32 types, can be found in [9].

However, it can be stated that, after renumbering the colors, the colorings all have one of the following four properties:

- at most four colors are used;
- after the deletion of three vertices, color 1 is dominant (see Theorem 2(b));

- color 1 is dominant and color 2 is subdominant; that is, the vertices can be partitioned into classes $U, V_3, V_4, V_5, \ldots$ such that edges meeting U are colored 1 or 2, the edges between classes V_i, $i \geq 3$ are colored 1 and edges within class V_i are colored 1, 2 or i;
- color 1 is nearly dominant; that is, the vertices can be partitioned into classes $W, V_5, V_6, V_7, \ldots$ such the edges between classes are all colored 1. Moreover the edges within V_i are colored 1 or i, and the edges within W are colored 1, 2, 3 or 4 in such a way that every rainbow 3-path within W contains an edge of color 1;

Lemma 3.. *Let K_n be edge colored so that it contains a rainbow 4-path P_5 and a vertex disjoint rainbow star with 5 edges. Then it contains a rainbow 5-path P_6.*

We define the color degree of a vertex to be the number of different colors appearing on its incident edges.

Notice that, in the colorings listed in Theorem , only (b) and (c) may use more than 4 colors, so any coloring containing no rainbow P_5 has at most one vertex of color degree 5 or greater.

Theorem 4.. *Let K_n be edge colored so that it contains no rainbow 5-path P_6. Then there are at most 3 vertices whose color degree is 10 or greater.*

Theorem 5.. *There exists a coloring of K_n containing no rainbow 6-path P_7 in which $\lceil n/2 \rceil$ vertices have color degree $1 + \lfloor n/2 \rfloor$.*

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Presented paper by Hossein Teimoori Faal

Clique Polynomials

Introduction

Throughout this paper we only consider simple graphs. By a simple graph $G = (V, E)$ we mean a finite undirected graph with no loops and multiple edges with vertex set $V = V(G)$ and edge set $E = E(G)$.
For every simple graph G, we assign a polynomial called *clique polynomial* [1], as follows

$$C(G; x) = 1 + c_1 x + c_2 x^2 + \cdots + c_\omega x^\omega,$$

where c_i, $i > 0$, is the number of cliques (complete subgraphs) of G on i vertices.

In this research, we mainly concentrate on the roots of the clique polynomials and it's connection with structural properties of underlying graphs. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G-v$ we mean


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the subgraph $G[V-v]$, if $v \in V(G)$. We denote by $G-uv$ the subgraph of G obtained by deleting the edge $e = uv$, for $e \in E(G)$. The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$. We also use the notations K_m, $K_{m,n}$ and P_n for *complete* graph, *complete bipartite* graph and *path*, respectively. For other definitions we refer reader to [2].

Examples.

1. $C(K_m; x) = (1+x)^m$,
2. $C(K_{m,n}; x) = (1+mx)(1+nx)$,
3. $C(P_m; x) = (1+x)(1+(m-1)x)$,
4. $C(C_m; x) = 1 + mx + mx^2$.

Note that all trees on n vertices have the same clique polynomial. Therefore, in general, a graph can not be determined uniquely by it's clique polynomial.

Clique Polynomials and Their Roots

We can construct the clique polynomial of a graph recursively, using two operations on graphs, that is, *deleting* a vertex or an edge. Indeed, we have the following result.

Lemma 1. *Let G be a graph and let $v \in V(G)$. Then*

1. $C(G; x) = C(G-v; x) + xC(G[N(v)]; x)$,
2. $C(G; x) = C(G-uv; x) + x^2 C(G[N(v) \cap N(v)]; x)$.

For the rest of the paper we need the following definition.

Definition 2. Let G be simple graph. We define *Z(G)* as the set of all nonnegative real roots of $C(G; x)$. If *Z(G)* is non-empty then define ζ_G to be the max*Z(G)* and otherwise to be $-\infty$.

In Lemma 2 we saw the connection between the clique polynomial of a graph and the clique polynomials of induced subgraphs resulting from deleting one of it's vertices. Now, the following natural question arise.

Question. What is the connection between the ζ_G and ζ_H, where H is the induced subgraph of G?

Before answering this question, we need to review the following important result from calculus due to *Bolzano* (see [3, page 143]).

Lemma 3.. *Let f be a continuous function at each point of a close interval $[a, b]$ and assume that that $f(a)$ and $f(b)$ have opposite signs. Then there is at least one c in the open interval (a, b) such that $f(c) = 0$.*
Now, the following theorem answer the above question.

Theorem 4. *If G is a graph and H is one of it's induced subgraph, then $\zeta_H \leq \zeta_G$.*

Proof. We proceed by induction on n. For $n = 1$ and $n = 2$ the assertion is obvious. If H is an arbitrary proper induced subgraph of G, then we can find a vertex v of G such H is the induced subgraph of $G - v$. Hence it is sufficient to prove the theorem for induced subgraphs of the form $G - v$, for some $v \in V(G)$ Now, assume that $v \in V(G)$. If $Z(G - v) = \emptyset$ then there is nothing to prove. So we can assume that $Z(G - v)$ is not empty. By part (1) of Lemma 2 we have

$$C(G; x) = C(G - v; x) + xC(G[N(v)]; x).$$

Substituting ζ_{G-v} in both sides of the above equation and applying inductive hypothesis, that is $\zeta_{G[N(v)]} \leq \zeta_{G-v}$, we have $C(G[N(v)]; \zeta_{G-v}) \geq 0$, thus $C(G; \zeta_{G-v}) \leq 0$. But we also have $C(G; 0) \leq 0$. Therefore, the proof is complete using Lemma 4. □
As an immediate corollary of the above theorem, we get the following result which can be viewed as a strong tool in the theory of *forbidden subgraphs*. Recall that the graph G is *H-free* if G does not contain the graph H as an induced subgraph.

Corollary 5. *If G and H are two graphs such that $\zeta_H > \zeta_G$, then G is a H-free graph.*
Now we are at the position to state our main result.

Theorem 6. *For every graph G, the clique polynomial of G, has always a real root. In particular, $-1 \leq \zeta_G < 0$.*

Proof. The proof is straight forward and left to the reader as a simple exercise. □
As an application of the above theorem, we give an elementary proof of Turan's graph theorem for *triangle-free* graphs.

Corollary 7. *If G is a triangle-free graph then $|E(G)| \leq |V(G)|^2/4$.*

Proof. Since G has no triangle, we have

$$C(G; x) = 1 + |V(G)|x + |E(G)|x^2.$$

By Theorem 6, $C(G;x)$ has a real root which implies that the discriminant of the above quadratic polynomial, that is $|V(G)|^2 - 4|E(G)|$, is nonnegative; as claimed. □

By cleverly choosing the special induced subgraphs of G, we can obtain the following results.

Corollary 8. *Let G be a graph and $\alpha(G)$ be the independence number of G. Then*

$$\alpha(G) \leq -1/\zeta_G.$$

Corollary 9. *Let G be a graph that is not complete and let $g(G)$ be the grith of G. Then*

$$g(G) \leq \frac{-1}{\zeta_G^2 + \zeta_G}.$$

Corollary 10. *Let G be a connected graph of diameter d. Then*

$$d \leq -1/\zeta_G.$$

Now, the second part of Lamma 2, suggest the following question.

Question. What is the connection between ζ_G and ζ_H, where H is the spanning subgraph of G?

In this direction we have the following theorem, which is obtained by applying the second part of Lemma 1 and Lemma 3.

Theorem 11. *If G is a graph and H is a spanning subgraph of G, then $\zeta_G \leq \zeta_H$.*

As an immediate consequence of the above theorem, we have the following result.

Corollary 12.. *Let G be a graph with n vertices, we have*

1. *If $n \geq 4$ and $\zeta_G > \frac{-1+\sqrt{1-4/n}}{2}$, then G is not Hamiltonian.*
2. *If $n \geq 2$ and $\zeta_G > -1 + \sqrt{1-2/n}$, then G does not have perfect matching.*

Remark. Using Theorem4 and Theorem11, we can refine the inequality

for ζ_G of a connected graph G, as follows

$$\frac{-1}{diam(G)} \leq \zeta_G \leq \frac{-1}{|V(G)|-1}$$

where diam(G) is the diameter of the connected graph G.

Some Problems and Questions for Further Research

1. Find all families of graphs which can be determined uniquely by their clique polynomials.
2. Determine all graphs G with $\zeta_G = \frac{-1}{m}$, where m is a positive integer number.
3. What is the combinatorial interpretation of the coefficients of the series expansion of $\frac{1}{C(G;-x)}$?
4. Determine the families of graphs having only real roots for their clique polynomials.
5. Is it possible to generalize the idea in Corollary 7, for giving an elementary proof of Turan's Graph Theorem for K_4-*free* graphs?

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Presented paper by Yahya Ould Hamidoune

On iterated image size for point-symmetric relations

(http://arxiv.org/abs/math.CO/0603504v1)

Introduction

The Cacceta and Häggkvist conjecture is based in a very easy idea: many arrows in a directed digraph will make short directed cycles.

The conjecture proposes an upper bound for the girth of the digraph, depending only on the number of vertices and the minimum degree of these. Seymour made, later, another conjecture, impliing the first one.

We are proving Seymour's conjecture for vertex-transitive digraphs.

More formally

Definition 1. The directed graph $\Gamma = (V, E)$ is called *vertex-transitive* if for all $x, y \in V$, there exists an automorphism f such that $y = f(x)$.

Conjecture 2. *(Cacceta-Häggkvist) Let Γ be a directed vertex-transitive graph with loops at every vertex and let $r = min\{|\Gamma(v)| : v \in V\}$. Then, $g \leq \left\lceil \frac{|V|}{r-1} \right\rceil$.*

Conjecture 3. *(Seymour) Let $\Gamma = (V, E)$ be a directed vertex-transitive graph with loops at every vertex and let $g \geq 2$ be an integer. Then one of the following conditions holds.*

- *There are $y \in V$ and i such that $2 \leq i \leq g-2$ and $\Gamma^-(y) \cap \Gamma^i(y) \neq \{y\}$.*
- *There is an $x \in V$ such that $|\Gamma^{g-2}(x)| \geq 1 + (g-2)(|\Gamma(x)| - 1)$.*

Lemma 4. *If Γ is vertex transitive and $a(\Gamma) \leq a(\Gamma^-)$ then, for each atom A, the subgraph induced by Γ on A, $\Gamma[A]$, is also vertex-transitive itself.*

Theorem 5. *Let $\Gamma = (V, E)$ be a directed vertex-transitive graph with loops at every vertex and let $v \in V$. Let $j \geq 1$ be such that $\Gamma^j(v) \cap \Gamma^-(v) = \{v\}$. Then $|\Gamma^j(v)| - |\Gamma^{j-1}(v)| \geq |\Gamma(v)| - 1$.*

Proof First, we easily can check that theorem holds for $\kappa(\Gamma) = |\Gamma(v)| - 1$.

If $\kappa(\Gamma) < |\Gamma(v)| - 1$, by lemma 4, we will be able to make the proof by induction, using the induction hypothesis when we work inside an atom.

We will first assume $a(\Gamma) \leq a(\Gamma^-)$, because we need it for the lemma. But if this does not hold, then the theorem can be proved for Γ^-, which is equally useful for us. □

Corollary 6. *This result proves the Seymour's conjecture for vertex-transitive graphs and thus, the Cacceta-Häggkvist conjecture too.*

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Presented paper by D. Piguet, M. Stein

An approximate version of the Loebl-Komlós-Sós Conjecture

Introduction

During the talk I shall first present the notion of regularity and state Szemerédi's Regularity Lemma (Theorem 3), which is a very powerful theorem in graph theory. As an illustration, I shall then present a result of Maya Stein and myself (Theorem 4) and give a short sketch of the proof.

More formally

Definition 1. A pair (X, Y) is (α, ε)-*regular* if, for any subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \alpha|X|$ and $|Y'| \geq \alpha|Y|$, we have $|\delta(X, Y) - \delta(X', Y')| < \varepsilon$, where δ denotes the density.

Observation 2. *If (X, Y) is a regular pair, then for all but at most $\alpha|X|$ vertices $v \in X$, we have $deg_Y(v) \geq deg_Y(X) - \varepsilon|Y|$, where $deg_Y(X)$ denotes the average degree of the set X into the set Y.*

Theorem 3. *For every $\varepsilon, \alpha > 0$ and $m \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ so*

that every graph G of order $n \geq n_0$ admits a partition of its vertex set $V(G) = V_0 \cup V_1 \cup \ldots \cup V_N$ with $m \leq N \leq M$ such that $|V_i| = |V_j|$ for $i, j \geq 1$, $|V_0| \leq \varepsilon n$, and all but at most εN^2 pairs (V_i, V_j) with $i, j \geq 1$ are (ε, α)-regular.

Theorem 4. *For every $\pi, q > 0$ there is an $n_0 \in \mathbb{N}$ so that for all graphs G on $n \geq n_0$ vertices the following is true. If at least $(1+\pi)\frac{n}{2}$ vertices of G have degree at least $(1+\pi)qn$, then G contains, as subgraphs, all trees with at most qn edges.*

Proof After using Regularity Lemma and deleting some undesirable edges, we get a so called *cluster graph*, where each edge represents a dense regular pair. Using Gallai-Edmonds Matching Theorem on this cluster graph, we get two possible configurations that catches in some way the structure of the graph G. In both configurations, we get two clusters A and B with high average degree, relatively high density between them, and a "nice" neighbourhood, depicted in two different cases.

In the first case, the neighbourhoods of clusters A and B are covered (in the corresponding cluster graph) by a matching. In the second case, the neighbourhood of A is covered by a matching, and a large portion of the neighbourhood of B is covered by a matching or clusters with high average degree.

Then, we partition the tree into subtrees of small size and embed those small subtrees one after the other using regular pairs represented by edges of the matching or edges containing a cluster with high average degree.
□

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Asymetrical Ramsey-type theorems for hypercubes

Introduction

In this talk I would like to introduce some new results from my diploma thesis. The subject falls within the Euclidean Ramsey theory, more precisely to the asymmetrical part of it. Euclidean Ramsey theory is examining konfigurations of points, for which there exists n such that for every coloring of n-dimensional Euclidean space with r colors we can find translated and/or rotaded copy (we say congruent copy) of our configuration in a single color. Asymetric branch deals with situations when we look for different configuration in every color. The main interest of this work is proving some asymetrical ramsey-type theorems for hypercubes, i.e. we will be coloring a n-dimensional Euclidean Space with several colors and searching for a monochromatic congruent copy of a d-dimensional unit hypercube in some color. The dimension of searched hypercube may vary depending on the color. We are going to prove some bounds on the dimension n of the Euclidean space with respect t o the number of colors and to the dimensions of searched hypercubes in these colors.

More formally

First of all we should define what are we going to bound:

Definition 1. Let $N(k_1, k_2, \dots k_r)$ the smallest possible dimension n such that for any coloring $\nu : E^n \to \{1, 2, \dots r\}$ there exists $i \in \{1, 2, \dots r\}$ such that there is unit hypercube of dimension k_i in color i. We will call such number $N(k_1, k_2, \dots k_r)$ *generalised Ramsey number* of respective hypercubes.

Now we can proof the upper bound for one class of special cases:

Theorem 2. $N(k, \underbrace{1, 1, \dots 1}_{l}) \leq l \cdot 2^{k-1} + k.$

As in many other cases in the Ramsey theory we can bound the generalised Ramsey number only by exponencial expression and only in these special cases by now. The proof is using mathematical induction and one simple geometrical construction which will however need some spatial imagination. In the rest of time I will show some colorings that will give us lower bounds for some special cases, namely that $N(2, 1, 1, 1, 1, 1) \geq 3$ and $N(2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1) \geq 4$.

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Presented paper by S. Jendrol', J. Miškuf and R. Soták

Presented paper by S. Brandt, J. Miškuf and D. Rautenbach

Recent Results on Edge Total Irregular Labellings

Introduction

We present a brief survey of results on the total edge irregularity strength. Most of there results can be found in [3], [8].

Chartrand et al., in [4], proposed the following problem: Assign positive integer labels to the edges of a connected graph of order at least 3 in such a way that the graph becomes irregular, i.e. the weights (label sums) at each vertex are distinct. What is the minimum value of the largest label over all such irregular assignments? This parameter of a graph G is well known as the *irregularity strength* of the graph G, $s(G)$. Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [5], [6], [9]. For recent results see the papers by Amar and Togni [1], Jacobson and Lehel [7], Nierhoff [8], [5]. Motivated by these papers and by a book of Wallis [12], Bača et al. [2] started to investigate the total edge irregularity strength of a graph G, an invariant analogous to the irregularity strength for total labellings.

For a graph $G = (V, E)$ we define a labelling $\nu : V \cup E \to \{1, 2, ..., k\}$ to be *a total edge irregular k-labelling* of the graph G if for every two different edges e and f of G one has $\varphi(e) \neq \varphi(e)$ where the weight of an edge $e = \{u, v\}$ in the labelling ν is $\varphi(e) = \nu(u) + \nu(v) + \nu(e)$. The minimum k for which the graph G has an edge irregular total k-labelling is called *the total edge irregularity strength* of G, $tes(G)$. Let us mention a result from [2] concerning lower bound on the total edge irregularity strength of a graph.

Theorem 1. *Let $G = (V, E)$ be a graph with $|E(G)|$ edges. Then $tes(G) \geq$*

$max\left\{\left\lceil\frac{\Delta+1}{2}\right\rceil,\left\lceil\frac{|E(G)|+2}{3}\right\rceil\right\}$.

There are classes of graphs where both values from previous theorem are attained, see [2]. The authors of [2] present also some other families of graphs for which they found the exact value of the total edge irregularity strength.

Next we mention an upper bound on the total edge irregularity strength of graph G, if we are able to find an independent set of vertices of a graph G with the special property. Let us remark that the independent set of vertices S is a set $S \subseteq V$ of vertices such that there is no edge $e = uv$ with $u \in S$ and $v \in S$.

Theorem 2. *Let $G = (V, E)$ be a graph with p vertices and q edges. Let $I = \{u_1, ..., u_n\}$ be an independent set of vertices of a graph G such that $\sum_{i=1}^{n} deg(u_i) \leq \frac{q-1}{2}$. Then*

$$tes(G) \leq q - \sum_{i=1}^{n} deg(u_i) \quad .$$

Recently Ivančo and Jendrol' [7] determined the total edge irregularity strength for any tree and posed the intriguing conjecture:

Conjecture 3. *Let $G = (V, E)$ be a graph different from K_5 with maximum degree Δ, then*

$$tes(G) \geq \max\left\{\left\lceil\frac{\Delta+1}{2}\right\rceil,\left\lceil\frac{|E(G)|+2}{3}\right\rceil\right\} \quad .$$

Main results

In the first part we present results due to S. Jendrol', J. Miškuf, R. Soták, see [8]. We have determined the total edge irregularity strength of complete graphs. Every graph G with vertex set $V(G)$ is a subgraph of a complete graph on the same vertex set $V(G)$. The total edge irregularity strength is a monotone graph invariant, i.e., if G is a subgraph of graph H then $tes(G) \leq tes(H)$. Hence we have next lemma.

Lemma 4. *Let $G = (V, E)$ be a graph with $|V(G)| = n$ vertices. Then $tes(G) \leq tes(K_n)$.*

We show that the total edge irregularity strength of almost all complete graphs is equal to the bound in Theorem 1. First we prove that the total edge irregularity strength of the K_5 is 5 but the lower bound is 4 (see [2]).

Observation 5. $tes(K_5) = 5$.

Proof Let us assume for a contradiction that $tes(K_5) = 4$. Since the weight of an edge is the sum of three natural numbers its minimum value is 3. Using the same argument observe that its maximum value is 12. As K_5 has ten edges there have to be ten distinct weights. Therefore all the possible weights between 3 and 12 have to appear. To obtain weight 3 there must be two vertices labelled with label 1 and to obtain weight 12 there must be two vertices labelled with label 4. Now we have four edges with one end-vertex labelled with 1 and the other labelled with 4. We have to use all four labels from 1 to 4 to guarantee diversity of weights. Clearly, the fifth vertex can be labelled neither with 1 nor with 4. If the fifth vertex would be labelled with label 2 then there is no possibility to obtain weights 10 and 11 but if the last vertex would be labelled with label 3 there is no possibility to obtain weights 3 and 4. □

Theorem 6. *Let $n \in \mathbb{N}$ and K_n be the complete graph on n vertices, $n \geq 3$, $n \neq 5$. Then $tes(K_n) = \left\lceil \frac{n^2-n+4}{6} \right\rceil$.*

Next we deal with complete bipartite graphs. The exact value of the total edge irregularity strength of complete bipartite graphs is found.

Theorem 7. *Let $n, m \in \mathbb{N}$, $n, m \geq 2$ and $K_{m,n}$ be a complete bipartite graph. The total edge irregularity strength of $K_{m,n}$ is $tes(K_{m,n}) = \left\lceil \frac{m \cdot n+2}{3} \right\rceil$.*

In the rest we present results due to S. Brandt, J. Miškuf, D. Rautenbach, see [3]. We proved that Conjecture 3, posed by J. Ivančo and S. Jendrol', holds when $\left\lceil \frac{\Delta+1}{2} \right\rceil \geq \left\lceil \frac{|E|+2}{3} \right\rceil$ for a graph G. Actually, we proved a stronger result.

Theorem 8. *If $G = (V, E)$ is a graph with maximum degree $\Delta \geq \frac{2|E|}{3}$ then it has a edge irregular total labelling with $k = \lceil \frac{\Delta+1}{2} \rceil$ labels.*

In the other case when $\left\lceil \frac{|E|+2}{3} \right\rceil \geq \lceil \frac{\Delta+1}{2} \rceil$ we showed that Conjecture 3 holds for graphs with bounded maximum degree Δ. More formally:

Theorem 9. *Every multigraph $G = (V, E)$ without loops of order n, size m and maximum degree $0 < \Delta < \frac{10^{-3}m}{\sqrt{8n}}$ satisfies*

$$tes(G) = \left\lceil \frac{|E| + 2}{3} \right\rceil \quad .$$

Moreover, we show that Conjecture 3 holds for dense graphs.

Corollary 10. *Every graph $G = (V, E)$ of order n and size $m \geq 3000n^{\frac{3}{2}}$ satisfies $tes(G) = \lceil \frac{m+2}{3} \rceil$.*

We also prove that for a fixed maximum degree Δ there should be only finitely many counterexamples to mentioned conjecture.

Corollary 11. *For every integer $\Delta \geq 1$ there is some $n(\Delta)$ such that every graph $G = (V, E)$ without isolated vertices with order $n \geq n(\Delta)$, size m and maximum degree at most Δ satisfies $tes(G) = \lceil \frac{m+2}{3} \rceil$.*

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Presented paper by Ken-ichi Kawarabayashi and Bruce Reed

Fractional Coloring and the odd Hadwiger's conjecture

Gerards and Seymour cojectured that if a graph has odd compelate minor od order p, then it is $(p-1)$-colorable. This is an analogue of the well-known conjecture of Hadwiger, and in fact, this would immediately imply Hadwiger's conjecture.

Motivated by this, we prove that the "fractional chromatic number" of a graf G without odd K_p-minor is at most $2p$; that is, it is possible to assign a rational $q(S) \geq 0$ to every stable set $S \subseteq V(G)$ so that $\sum_{S \ni v} q(S) = 1$ for every vertex v, and $\sum_S q(S) \leq 2p$.

This generalizes the result by Reed and Seymour who proved that the fractional chromatic number od a graph with no K_{p+1}-minor is at most $2p$.

Definition 1. We say that H has *odd complete minor* of size at least p if there are p vertex disjoint trees in H such that every two of them are joined by an edge, and in addition, all the vertices of trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic. We say that H has an *odd K_l-minor*, if H has an odd complete minor of size at least l.

Formally we prove in this talk this theorem.

Theorem 2. *For every integer $p \geq 1$, every graph with no odd K_p minor has a fractional $2p$-colouring.*

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Counting polygon dissections in the projective plane

Abstract

For each value $k \geq 2$, we determine the number p_n of ways of dissecting a polygon in the projective plane into n subpolygons with $k+1$ sides for each. In particular, if $k = 2$ we recover a result of Edelman and Reiner (1997) on the number of triangulations of the Moebius band having n labelled point on its boundary. We also solve the problem when the polygon is dissected into subpolygons of arbitrary size. In each case, the associated generating function $\sum p_n z^n$ is a rational function in z and the corresponding generating function of plane polygon dissections. We also present some related problems.

Triangulations over the projective plane

The presentation will be focussed to explain how to use the symbolic method [1] and some topology tools (fundamental group of a surface and homology groups) to prove this result:

Theorem 1. *Let p_n be the number of non-plane triangulations (i.e., simplicial decompositions) of a polygon with n vertices in the projective plane, $P(z)$ the corresponding generating function and let $C(z)$ be the generating function for the Catalan numbers. Then:*

$$P(z) = \frac{2 - 9z + 6z^2 + 7z^3 - 2z^4}{z(1-4z)} C(z) - \frac{2 - 7z + z^2 + 5z^3}{z(1-4z)} = \tag{44}$$
$$= z^5 + 14z^6 + 113z^7 + 720z^8 + \dots$$

This sequence appears in Sloane's *On-Line Encyclopedia of Integer Sequences* as $A007817$. This is a known result [1], but our approach only uses the language of generating functions.

The proof of this theorem needs a combinatorial decomposition of the problem, which can be reduced into a topological problem. After finding this decomposition, we will count planar triangulation with some restrictions (non-allowable diagonals).

Other constructions

Combinatorial decomposition used in the enumeration of triangulations can be used in several families of dissections [2]. In our study we have obtained explicit expressions for:

1. Decompositions of the projective plane into polygons with $k+1$ vertices ($k > 2$).
2. General dissections.

In all cases, the ordinary generating function obtained is a polynomial expression with coefficients in $\mathbb{C}(z)$ on the corresponding planar generating function.

Using transfer theorem's [3], it is a simple exercise to deduce the asymptotic behavior of such constructions.

Further work

Those type of decompositions can be extended to constructions in every compact and connected surface. It is natural to ask if the corresponding generating functions will be written as a polynomial of the corresponding planar generating function, as it happened in the projective plane.

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Volumes in High Dimension

Introduction

This talk covers part of second chapter from Jiří Matoušek's text prepared from Spring School [1].

First we say a few words about volume of n-dimensional convex bodies (we compare the volume of the n-dimensional cube with the volume of the unit ball inscribed in it) and about some paradoxes, when bodies behave quite differently in high dimensions from what the 3-dimensional intuition suggests.

Then we introduce a generally useful tool – nets in a sphere.

Finally we devote ourselves to aproximating convex bodies by ellipsoids. Broadly speaking, each symmetric convex body looks like an ellipsoid.

More formally

Definition 1. $N \subseteq S^{n-1}$ (unit sphere in $\mathbb{R}^n$) is an η-net if it is an inclusion-maximal η-separated subset of S^{n-1}, whereas η-separated means that every two distinct points of N have (Euclidean) distance greater than η.

Lemma 2. *(Size of η-nets in the sphere). For each $\eta \in (0, 1]$, any η-net $N \subseteq S^{n-1}$ satisfies*

$$|N| \leq \left(\frac{4}{\eta}\right)^n.$$

Theorem 3. *(John's Lemma) Let $K \subset \mathbb{R}^n$ be a bounded closed convex body with nonempty interior. Then there exists an ellipsoid E such that*

$$\frac{1}{n}E \subseteq K \subseteq E.$$

If K is symmetric about the origin, then we have the improved aproximation

$$\frac{1}{\sqrt{n}}E \subseteq K \subseteq E.$$

References

[1] J. Matoušek: Combinatorial Excursions to High-Dimensional Convex Geometry, *KAM-DIMATIA Series 2001-515*

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Presented paper by D. Meister, J.A. Telle and M. Vatshelle

Digraphs of bounded Kelly-width

(http://www.ii.uib.no/∼telle/bib/MTV.pdf)

Introduction

The tractability of large classes of NP-complete problems when parameterized by the treewidth of the input graph counts among the strongest results in algorithmic graph theory. The algorithms behind this tractability have two stages: first an algorithm computing treewidth, then an algorithm solving the specific problem using the tree-structure discovered in the first stage.

Kelly-width is a parameter of directed graphs recently proposed by Hunter and Kreutzer [1] as a directed analogue of treewidth. We give several alternative characterizations of directed graphs of bounded Kelly-width in support of this analogy. We apply these results to give the first polynomial-time algorithm recognizing directed graphs of Kelly-width 2.

More formally

Definition 1. Let $k \geq 0$. The class of *generalised k-DAGs*, k-GDAGs, for short, is the class of digraphs inductively defined by the two following construction steps:

(1) a graph on one vertex is a k-GDAG

(2) let G be a k-GDAG and let u be a vertex that does not appear in G. Let X be a set of at most k vertices of G, called the *parent vertices of* u. Then, G' is a k-GDAG where G' emerges from G by adding vertex u and the following arc set:

$$\left\{(u,x) : x \in X\right\} \cup \left\{(y,u) : y \in \bigcap N_G^{\text{in}}[X]\right\}.$$

Definition 2. Let $k \geq 0$, and let G be a digraph. G is a *partial k-GDAG* if and only if G is a partial graph of some k-GDAG.

Theorem 3. *Let $k \geq 0$, and let G be a digraph. G has Kelly-width at most $k+1$ if and only if G is a partial k-GDAG.*

Theorem 4. *Let $k \geq 0$. An undirected graph G is a chordal graph of treewidth at most k if and only if there is a k-GDAG H such that $G =$ bi-dir(H).*

Definition 5. Let $G = (V, A)$ be a digraph on at least two vertices, and let x be a vertex of G. The operation *reducing G by x* yields graph G' that is obtained from G by deleting vertex x and adding the arc set $\{(u,v) : u \neq v$ and $u \in N_G^{\text{in}}(x)$ and $v \in N_G^{\text{out}}(x)\}$.

Theorem 6.*[1] Let $k \geq 0$, and let $G = (V, A)$ be a digraph. Then, G has Kelly-width at most $k+1$ if and only if G can be reduced to a graph on one vertex by repeatedly reducing by a vertex of out-degree at most k.*

This theorem implies an easy algorithm for recognizing graphs of bounded Kelly-width. Unfortunately, this algorithm is not a polynomial-time algorithm. A given graph, partial k-GDAG or not, can have more than one vertex of out-degree at most $k-1$. There is no a priori argument or criterion deciding which one to choose.

Theorem 7. *Let $G = (V, A)$ be a digraph, and let a be a vertex of out-degree at most 1 of G. Then, G is a partial 1-GDAG if and only if the graph obtained from reducing G by a is a partial 1-GDAG.*

This theorem immediately gives us an algorithm for recognizing digraphs of kellywidth at most 2. Repeatedly reduce by a vertex until either there is only 1 vertex, then answer yes, or there is no vertex of out-degree at most 1.

Open problems

(1) Is Kelly-width the best directed analogue for treewidth?

(2) Recognition algorithms for values larger than 2?

(3) Which problems can be solved with kellywidth as a parameter?

References

[1] P. Hunter and St. Kreutzer: Digraph Measures: Kelly Decompositions, Games, and Orderings., *In Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 637644*, SODA, 2007.

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Presented paper by Sandi Klavžar, Simon Špacapan, Janez Žerovnik

Perfect codes in direct products of cycles

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Abstract

Let $G = \prod_{i=1}^{n} \mathcal{C}_{l_i}$ be a direct product of cycles. It is proved that for any $r \geq 1$, and any $n \geq 2$, each connected component of G contains an r-perfect code provided that each l_i is a multiple of $r^n + (r+1)^n$. On the other hand, if a code of G contains a given vertex and its canonical local vertices, then any l_i is a multiple of $r^n + (r+1)^n$. It is also proved that an r-perfect code ($r \geq 2$) of G is uniquely determined by n vertices, and it is conjectured that for $r \geq 2$ no other codes in G exist other than the constructed ones.

Preliminaries

For a graph G the distance $d_G(u, v)$, or briefly $d(u, v)$, between vertices u and v, is defined as the number of edges on a shortest u, v-path.

Definition 1. A set $C \subseteq V(G)$ is an *r-code* in G if $d(u, v) \geq 2r + 1$ for any two distinct vertices $u, v \in C$. The code C is called an *r-perfect code* if for any $u \in V(G)$ there is exactly one $v \in C$ such that $d(u, v) \leq r$.

Note that C is an 1-perfect code if and only if the closed neighborhoods of its elements form a partition of $V(G)$.

Definition 2. For $u \in V(G)$ and $r \geq 0$ we call the set $B(u,r) = \{v \in V(G) | d_G(u,v) \leq r\}$ the *r-ball with center u*, r is called *radius*.

In this terminology $C \subseteq V(G)$ is an r-perfect code if and only if the r-balls $B(u,r), u \in C$, form a partition of $V(G)$.

Definition 3. The *direct product* $G \times H$ of graphs G and H is the graph defined on the Cartesian product of the vertex sets of the factors. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$.

The direct product of graphs is commutative and associative in a natural way. Hence, for graphs $G_1, \ldots, G_n$ we may write

$$G = G_1 \times \ldots \times G_n = \prod_{i=1}^{n} G_i$$

without parentheses, and the vertices of G can be represented as vectors $v = (v_1, \ldots, v_n)$, where $v_i \in V(G_i), 1 \leq i \leq n$. For $G = \prod_{i=1}^{n} G_i$ we will use d_i to denote the distance function in the ith factor graph G_i , that is, $d_i = d_{G_i}$.

$G = \prod_{i=1}^{n} \mathcal{C}_{l_i}$ consists of 2^{k-1} isomorphic connected components, where k is the number of l_i s that are even.

Since G is vertex transitive, we will often implicitly, without loss of generality, assume that $v = \mathbf{0}$ is a fixed arbitrary vertex of the product considered.

For the cycle $C_k (k \geq 3)$ we will always assume $V(C_k) = \{0, 1, \cdots, k-1\}$. Whenever applicable, the computations will be done modulo k, that is, modulo the length of the appropriate cycle.

Results

Theorem 4. *Let $r \geq 1, n \geq 2$, and $G = \prod_{i=1}^{n} \mathcal{C}_{l_i}$, where each l_i is a multiple of $l = r^n + (r+1)^n$. Then each connected component of G contains an r-perfect code.*

Define $s = 2r + 1$.

In the proof the following vertices play a crucial role:

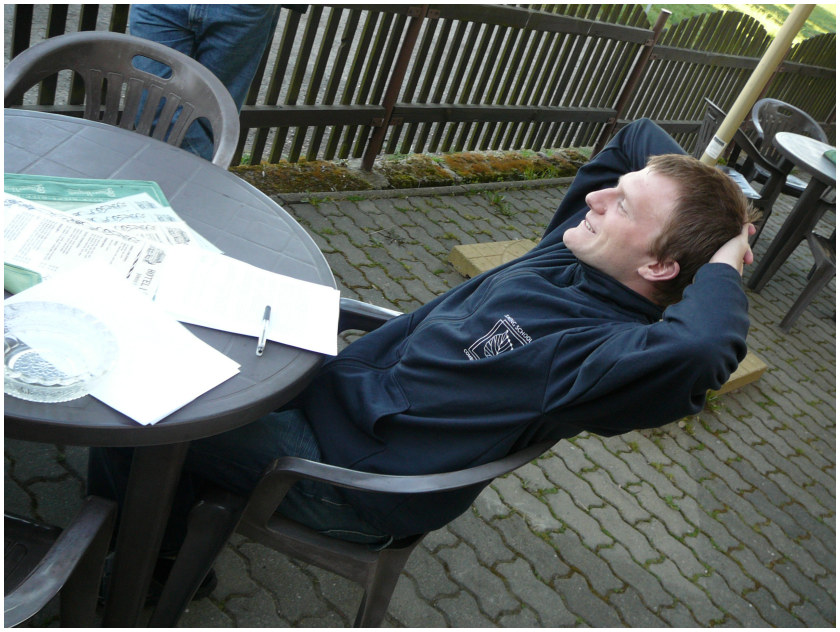
$$\begin{array}{rcl}
\mathbf{b}^1 & = & (\ \ s, \ \ 1, \ \ 1, \ \ldots, \ 1), \\
\mathbf{b}^2 & = & (-1, \ \ s, \ \ 1, \ \ldots, \ 1), \\
\mathbf{b}^3 & = & (-1, -1, \ \ s, \ \ldots, \ 1), \\
\vdots & & \qquad\qquad\qquad\quad \ddots \\
\mathbf{b}^n & = & (-1, -1, -1, \ \ldots, \ s),
\end{array}$$

Let us call the vertices $\mathbf{b}^1, \ldots, \mathbf{b}^n$ *canonical local vertices* for $\mathbf{0}$.

Theorem 5. *Let P be an r-perfect code of a connected component of $G = \prod_{i=1}^{n} \mathcal{C}_{l_i}$, where $r \geq 2, n \geq 2$, and $l_i \geq 2r+2$. Suppose that P contains $\mathbf{0}$ and the canonical local vertices for $\mathbf{0}$. Then every l_i is a multiple of $r^n + (r+1)^n$.*

Theorem 6. *Let $r \geq 2$ and $n \geq 3$. Then (a connected component of) $G = \prod_{i=1}^{n} \mathcal{C}_{l_i}, l_i \geq 2r+2$, contains an r-perfect code if and only if every l_i is a multiple of $r^n + (r+1)^n$.*

Stated for $n \in \{3, 4\}$ in paper, conjectured for greater n. Authors claimed in a personal communication, that they already proved the conjecture.



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