5-choosability of graphs with crossings far apart*

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Abstract

We give a new proof of the fact that every planar graph is 5-choosable, and use it to show that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable. At the same time we may allow some vertices to have lists of size four only, as long as they are far apart and far from the crossings.

Thomassen [5] gave a strikingly beautiful proof that every planar graph is 5-choosable. To show this claim, he proved the following more general statement:

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Theorem 1. *Let G be a plane graph with the outer face F, xy an edge of F, and L a list assignment such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus \{x, y\}$, $|L(x)| = |L(y)| = 1$ and $L(x) \neq L(y)$. Then G is L-colorable.*

Let us note that the lists of x and y of size 1 give a precoloring of a path of length 1 in the outer face of G. Unfortunately, this statement is rather tight, and almost any attempt to alter it (e.g., by allowing more than two vertices to be precolored, allowing lists of size 2 subject to some constraints, allowing some crossings in the drawing, etc.) fails with infinitely many counterexamples. We give a proof of a different version of Theorem 1 (see Theorem 6), that turns out to be more robust with respect to some strengthenings of the planar 5-choosability theorem. Our proof is inspired by Thomassen's proof [6] of 3-choosability of planar graphs of girth 5. Using this technique, we give the proof of our main result:

Theorem 2. *Let G be a graph drawn in the plane with some crossings and let $N \subseteq V(G)$ be a set of vertices such that the distance between any pair of crossed edges is at least 15, the distance between any crossing and a vertex in N is at least 13, and the distance between any two vertices in N is at least 11. Then G is L-colorable for any list assignment L such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.*

Let us recall that a *list assignment* L for G is a function that assigns to each vertex of G a set $L(v)$, called the *list of admissible colors* for v. An *L-coloring* is a choice of a color $c(v) \in L(v)$ for each $v \in V(G)$ such that no two adjacent vertices receive the same color. The graph is *k-choosable* if it admits an L-coloring for every list assignment L with $|L(v)| \geq k$ for every $v \in V(G)$.

Our main result, Theorem 2, clearly implies the following special case.

Theorem 3. *Every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable.*

Some distance condition on the crossings is necessary, even if we would allow only three crossings, as shown by K_6. On the other hand, it was proved in [4] and independently also in [2] that the distance requirement is not needed, if we have at most two crossings. The inductive proof of Theorem 2 involves a stronger inductive hypothesis that is stated later as Theorem 7 and in particular also implies the above-mentioned result from [2, 4].

Theorem 4 ([4, 2])**.** *Every graph whose crossing number is at most two is* 5*-choosable.*

The proof of Theorem 4 is given at the end of the paper. Another special case of Theorem 2 is the following.

Theorem 5. *Let G be a planar graph and $N \subseteq V(G)$ a set of vertices such that the distance between any pair of vertices in N is at least* 11*. Then G is L-colorable for any list assignment L such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.*

The last result is motivated by the result of Voigt [7] showing that not all planar graphs are 4-choosable. Furthermore, it is related to the following problem of Albertson [1]:

Problem 1. *Does there exist a constant d such that whenever G is a planar graph with list assignment L that gives a list of size one or five to each vertex and the distance between any pair of vertices with list of size one is at least d, then G is L-colorable?*

Starting with a similar technique as used in this paper, we were able to give a positive answer to this problem (which we present in a separate paper [3]).

We start with giving the proof of the basic statement for planar graphs in Section 1. In Section 2 we then generalize it by allowing crossings and 4-lists subject to distance constraints, obtaining the proof of Theorem 2.

1 Planar graphs

Let P be a path or a cycle. The *length* $\ell(P)$ of P is the number of its edges, i.e., a path of length l has $l+1$ vertices and a cycle of length l has l vertices. Given a graph G and a cycle $K \subseteq G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. For an integer $k \geq 2$, a path $v_0v_1 \ldots v_k$ is a k-*chord* if $v_0, v_k \in K$ and $v_1, \ldots, v_{k-1} \notin V(K)$. We define a chord to be a 1-*chord*. If G is a plane graph, let $\mathrm{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K, and $\mathrm{Ext}_K(G)$ the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K. In particular, $K = \mathrm{Int}_K(G) \cap \mathrm{Ext}_K(G)$. Note that each k-chord of K belongs to exactly one of $\mathrm{Int}_K(G)$ and $\mathrm{Ext}_K(G)$. If the cycle K is the outer face of G and Q is a k-chord of K, let C_1 and C_2 be the two cycles in $K \cup Q$ that contain Q. Then the subgraphs $G_1 = \mathrm{Int}_{C_1}(G)$ and $G_2 = \mathrm{Int}_{C_2}(G)$ are the Q-*components* of G.

As we have mentioned earlier, Thomassen's Theorem 1 does not extend to the case when we have a precolored path of length two. However, if we strengthen the condition on the list sizes of the other vertices on the outer face, such an extension is possible.

Theorem 6. *Let G be a plane graph with the outer face F, P a subpath of F of length at most two and L a list assignment such that the following conditions are satisfied:*

(i) $|L(v)| \geq 5$ *for* $v \in V(G) \setminus V(F)$,

(ii) $|L(v)| \geq 3$ *for* $v \in V(F) \setminus V(P)$,

(iii) $|L(v)| = 1$ *for* $v \in V(P)$,

(iv) *no two vertices with lists of size three are adjacent in G,*

(v) *L gives a proper coloring to the subgraph induced by $V(P)$, and*

(vi) *if $P = uvw$ has length two and x is a common neighbor of u, v and w, then $L(x) \neq L(u) \cup L(v) \cup L(w)$.*

Then G is L-colorable.

Proof. Suppose for a contradiction that the claim is false, and let G be a counterexample with $|V(G)|+|E(G)|$ the smallest possible, and subject to that, with the longest path P and with the minimum size of the lists (while satisfying (i)–(vi)). It is clear that G is connected and that every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.

Furthermore, G is 2-connected: otherwise, let v be a cut-vertex and let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{v\}$ and $|V(G_1)|, |V(G_2)| > 1$. If $v \in V(P)$, then by the minimality of G there exist L-colorings of G_1 and G_2, and these colorings together give an L-coloring of G. Otherwise, we may assume by symmetry that $P \subseteq G_1$. Consider an L-coloring φ of G_1. Let L_2 be the list assignment for G_2 such that $L_2(u) = L(u)$ for $u \neq v$ and $L_2(v) = \{\varphi(v)\}$. By the minimality of G, G_2 is L_2-colorable, and this coloring together with φ gives an L-coloring of G.

Every triangle T in G bounds a face: otherwise, first color the subgraph $\mathrm{Ext}_T(G)$ and then extend the coloring to $\mathrm{Int}_T(G)$. A similar argument shows that G contains no separating 4-cycles; otherwise, consider such a 4-cycle $K = k_1k_2k_3k_4$, and let φ be an L-coloring of $\mathrm{Ext}_K(G)$. Let $G' = \mathrm{Int}_K(G)$. Since K is separating, we have $V(G') \neq V(K)$, and since every triangle bounds a face, we conclude that K has no chord in G'. Let L' be the list assignment for $G' - k_1$ such that $L'(z) = \{\varphi(z)\}$ for $z \in \{k_2, k_3, k_4\}$, $L'(z) = L(z) \setminus \{\varphi(k_1)\}$ if $z \notin \{k_2, k_4\}$ is a neighbor of k_1 and $L'(z) = L(z)$ if z is any other vertex. By the minimality of G, the graph $G' - k_1$ is L'-colorable, and this coloring together with φ gives an L-coloring of G.

Since G is 2-connected, its outer face is bounded by a cycle, which we denote by F as well. Next, we show that F has no chords. Otherwise, let uv be a chord of F and let G_1 and G_2 be the uv-components of G. If $P \subseteq G_1$, then we first color G_1 and then extend the coloring to G_2. The case that $P \subseteq G_2$ is symmetric. It follows that P has length two and all the chords of F are incident with its middle vertex. Let $P = z_1uz_2$, where $z_i \in V(G_i)$ for $i \in \{1, 2\}$. Let φ be an L-coloring of G_1 and let L_2 be the list assignment for G_2 such that $L_2(z) = L(z)$ for $z \neq v$ and $L_2(v) = \{\varphi(v)\}$. Since G is not L-colorable, G_2 is not L_2-colorable. By the minimality of G, either

v is adjacent to z_2, or u, v and z_2 have a common neighbor w with list of size three (which means, in particular, that $w \in V(F)$). Since every chord of G is incident with u, the edge vz_2 or vw belongs to F. Since every triangle bounds a face, we conclude that v has degree two in G_2. By symmetry, v has degree two in G_1 as well, and thus v has degree three in G. It follows that $|L(v)| = 3$, and thus v cannot be adjacent to any other vertex with list of size three. In particular, we cannot have the case with the vertex w. We conclude that v is adjacent to z_1 and z_2 and $V(G) = \{u, v, z_1, z_2\}$. However, $L(v) \neq L(u) \cup L(z_1) \cup L(z_2)$ by (vi), and thus G is L-colorable. This contradiction proves that F has no chords.

Similarly, we have the following property:

(1) *Let uvw be a 2-chord of F and let G_1 and G_2 be uvw-components of G. If $P \subseteq G_1$, then either u and w are adjacent and G_2 is equal to the triangle uvw, or there exists a vertex x such that $V(G_2) = \{u, v, w, x\}$, $|L(x)| = 3$ and x is adjacent to u, v and w.*

If $\ell(P) < 2$, then it is easy to see that we can precolor $2 - \ell(P)$ more vertices of F without violating (vi). Thus, we may assume that $\ell(P) = 2$. Let $P = p_0p_1p_2$. Suppose that p_0, p_1 and p_2 have a common neighbor v. If $v \in V(F)$, then $V(G) = \{p_0, p_1, p_2, v\}$ and G is L-colorable. If $v \notin V(F)$, then v has degree at most four in G by (1) and thus $\deg(v) < |L(v)|$, which is a contradiction. Therefore, p_0, p_1 and p_2 have no common neighbor.

Furthermore, $\ell(F) \geq 6$: If $\ell(F) = 3$, then we remove one vertex of F and remove its color from the lists of all its neighbors, and observe that the resulting graph is a smaller counterexample to Theorem 6. In the case when $\ell(F) = 4$, then similarly color and remove the vertex of $V(F) \setminus V(P)$. Finally, suppose that $\ell(F) = 5$. Let φ be an arbitrary L-coloring of $F = p_2p_1p_0v_1v_2$. Remove v_1 and v_2 from G and remove their colors according to φ from the lists of their neighbors, obtaining a graph G' with the list assignment L'. Since every triangle in G bounds a face, at most one vertex in G' has list of size three. Since p_0, p_1 and p_2 have no common neighbor and p_0 is not adjacent to p_2, G' with the list assignment L' is a smaller counterexample to Theorem 6, which is a contradiction.

Let $F = p_2p_1p_0v_1v_2v_3v_4\ldots$. If $\ell(F) = 6$, then we set $v_4 = p_2$. We may assume that $|L(v_1)| = 3$ or $|L(v_2)| = 3$, since otherwise we can remove a color from the list of v_1. Let us consider a set $X \subseteq V(F) \setminus V(P)$ and a partial L-coloring φ of X that are defined as follows:

(X1) If $|L(v_1)| = 3$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ is chosen arbitrarily.

(X2) If $|L(v_1)| = 3$ and $|L(v_3)| = 3$, then $X = \{v_1, v_2\}$ and φ is chosen so that $\varphi(v_2) \in L(v_2)\setminus L(v_3)$ and $\varphi(v_1) \in L(v_1)\setminus(L(p_0)\cup \{\varphi(v_2)\})$.

(X3) If $|L(v_2)| = 3$, and either $|L(v_4)| \neq 3$ or $|L(v_3)| \geq 5$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2)$ is chosen arbitrarily.

(X4) If $|L(v_2)| = 3$, $|L(v_3)| = 4$ and $|L(v_4)| = 3$, then:

- (X4a) If v_1, v_2 and v_3 do not have a common neighbor or $|L(v_1)| \geq 5$, then $X = \{v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$ and $\varphi(v_2) \in L(v_2) \setminus \{\varphi(v_3)\}$.
- (X4b) If v_1, v_2 and v_3 have a common neighbor and $|L(v_1)| = 4$, then $X = \{v_1, v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3)\setminus L(v_4)$, $\varphi(v_1) \in L(v_1)\setminus L(p_0)$ and either at least one of $\varphi(v_1)$ and $\varphi(v_3)$ does not belong to $L(v_2)$, or $\varphi(v_1) = \varphi(v_3)$. The vertex v_2 is left uncolored.

For later reference, Figure 1 shows the subcases used in the definition of X and φ.

Let $G' = G - X$ and let L' be the list assignment obtained from L by removing the colors of the vertices of X according to φ from the lists of their neighbors (if a vertex of X is not colored according to φ, we do not remove any colors for it). Observe that any L'-coloring of G' can be extended to an L-coloring of G, thus G' is not L'-colorable. By the minimality of G, this implies that G' violates the assumptions of Theorem 6. Since F has no chords, the choice of

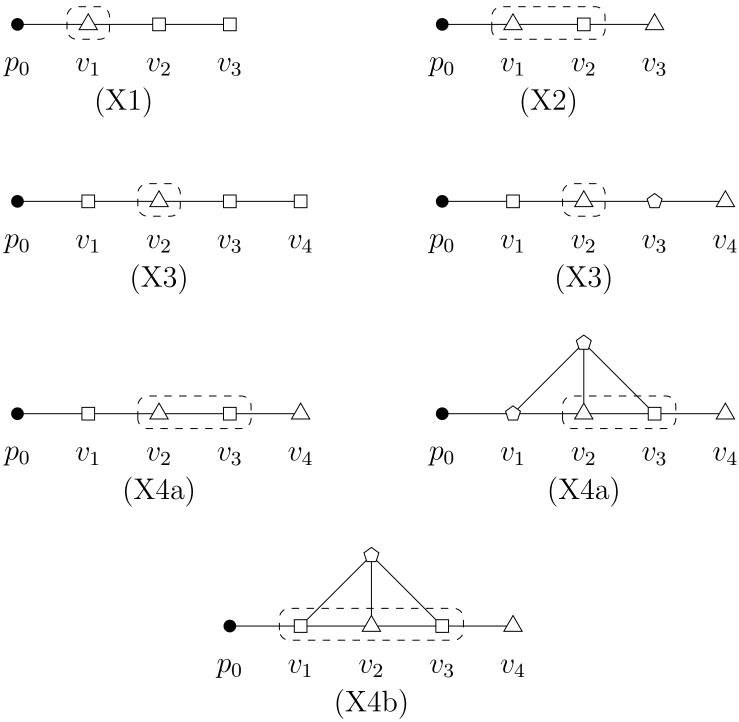


Figure 1: Subcases in the definition of X. Triangle vertices represent lists of size 3, square vertices list of size ≥ 4. Encircled vertices are in X.

X and φ implies that every vertex of $V(G') \setminus V(P)$ has list of size at least three. Since p_0 is not adjacent to p_2 and p_0, p_1 and p_2 do not have a common neighbor in G, it follows that the conditions (v) and (vi) are satisfied by G' with the list assignment L'. We conclude that (iv) is false, i.e., G' contains adjacent vertices u and v such that $|L'(u)| = |L'(v)| = 3$.

Since F has no chords, the choice of X ensures that at most one of u and v belongs to $V(F)$; hence, we can assume that $v \notin V(F)$ and v has two neighbors in X. In particular, X was chosen according to the cases (X2) or (X4). Since G contains no separating cycles of length at most 4, we conclude that u has at most one neighbor in X, and thus $u \in V(F)$. Let $x \in X$ be the neighbor of v such that the distance between u and x in $F - P$ is as large as possible. By (1) applied to the 2-chord xvu, we conclude that the xvu-component of G that does not contain P consists of xvu and a vertex z adjacent to x, v and u with $|L(z)| = 3$. It follows that $|L(u)| > 3$, and since $|L'(u)| = 3$, we have $z \in X$ and $|L(u)| = 4$. The inspection of the choice of X shows that (X4) holds, i.e., $u = v_1$, $z = v_2$ and $x = v_3$. However, note that the condition of (X4b) holds; hence $u \in X$, contrary to the assumption that $u \in V(G')$. This completes the proof of Theorem 6. □

2 Near-planar graphs

In this section, we aim to show that graphs drawn in the plane with crossings far apart are 5-choosable. For the purposes of the induction, it will be useful to allow other kinds of irregularities (adjacent vertices with list of size three, as well as vertices with list of size four not incident with the outer face, which arise when some vertices incident with a crossing are colored and their color is removed from the lists of their neighbors), subject to distance constraints.

Let us first introduce some terminology. Let G be a graph. For two subgraphs $H_1, H_2 \subseteq G$, the *distance* $d(H_1, H_2)$ between H_1 and H_2 is the minimum of the distances between the vertices of H_1 and

H_2, i.e., the minimum k such that there exists a path $v_0v_1 \ldots v_k$ in G with $v_0 \in V(H_1)$ and $v_k \in V(H_2)$. A *drawing* $\mathcal{G}$ of G in the plane consists of a set $\mathcal{V} = \{p_v \mid v \in V(G)\}$ of distinct points in the plane and a set of simple polygonal curves $\mathcal{E} = \{c_e \mid e \in E(G)\}$ such that

- if $uv \in E(G)$, then p_u and p_v are the endpoints of c_{uv},
- no internal point of any $c_e \in \mathcal{E}$ belongs to $\mathcal{V}$, and
- at most two of the curves in $\mathcal{E}$ contain any point that does not belong to $\mathcal{V}$, and any two curves in $\mathcal{E}$ have at most one point in common.

A *crossing* of $\mathcal{G}$ is a point in the plane that belongs to two of the curves in $\mathcal{E}$, but not to $\mathcal{V}$. An edge e is *incident with the crossing* x if $x \in c_e$. An edge e is *crossed* if it is incident with some crossing, and *non-crossed* otherwise. For a crossing x, we define G_x to be the graph consisting of the two edges incident with x. Two vertices of G are *crossing-adjacent* if they belong to G_x for some crossing x and are not adjacent in G_x. Removal of $\bigcup \mathcal{E}$ splits the plane into several connected subsets, which we call *faces* of $\mathcal{G}$. By a slight abuse of terminology, we sometimes identify a face with its boundary and hence speak about the vertices, edges and crossings of the face.

Let $\mathcal{G}$ be a drawing of a graph G, let P be a path of length at most three contained in the boundary of the outer face F of $\mathcal{G}$ (where in particular, no edge of P is crossed), N a subset of $V(G)$ and M a subset of $E(G)$, and let L be a list assignment for G. We say that L is (P, N, M)*-valid* if the following conditions are satisfied:

(S) $|L(v)| \geq 5$ for $v \in V(G) \setminus (V(F) \cup N)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$ and $|L(v)| = 1$ for $v \in V(P)$,

(N) $|L(v)| \geq 4$ for $v \in N \setminus V(F)$,

(M) if $|L(u)| = |L(v)| = 3$ and u and v are adjacent, then $uv \in M$,

(P) L gives a proper coloring to the subgraph induced by $V(P)$,

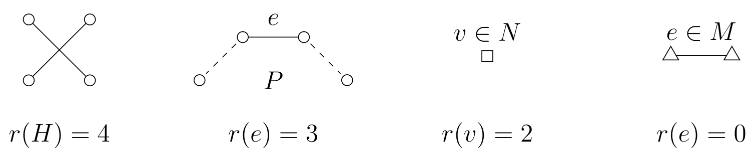


Figure 2: Special subgraphs and their ranks

(T) if a vertex v has three neighbors w_1, w_2, w_3 in $V(P)$, then $L(v) \neq L(w_1) \cup L(w_2) \cup L(w_3)$, and

(C) if x is a crossing and G_x contains a vertex with list of size three, then all other vertices of G_x have lists of size 1 or ≥ 5.

We define some subgraphs H of G to be *special*, and assign a *rank* $r(H)$ to each such subgraph (see Figure 2). Specifically, H is *special* if it falls into one of the following cases:

- H consists of the two edges incident with a crossing. In this case, its rank is 4.
- P has length three and H consists of the middle edge of P; the rank of H is 3.
- H is equal to a vertex of N, and $r(H) = 2$.
- H is equal to an edge of M, and $r(H) = 0$.

The drawing $\mathcal{G}$ is (P, N, M)-*distant* if $d(H_1, H_2) \geq r(H_1) + r(H_2) + 7$ for every pair $H_1 \neq H_2$ of special subgraphs of G. We shall occasionally refer to the (P, N, M)-distant requirement as the *distance condition*. The purpose of the introduced rank function is the following. In our inductive arguments, we will occasionally construct a smaller graph G' and introduce a new special subgraph H' in a vicinity of a special subgraph H that would no longer exist in G'. If H' has smaller rank than H, the distance condition for special

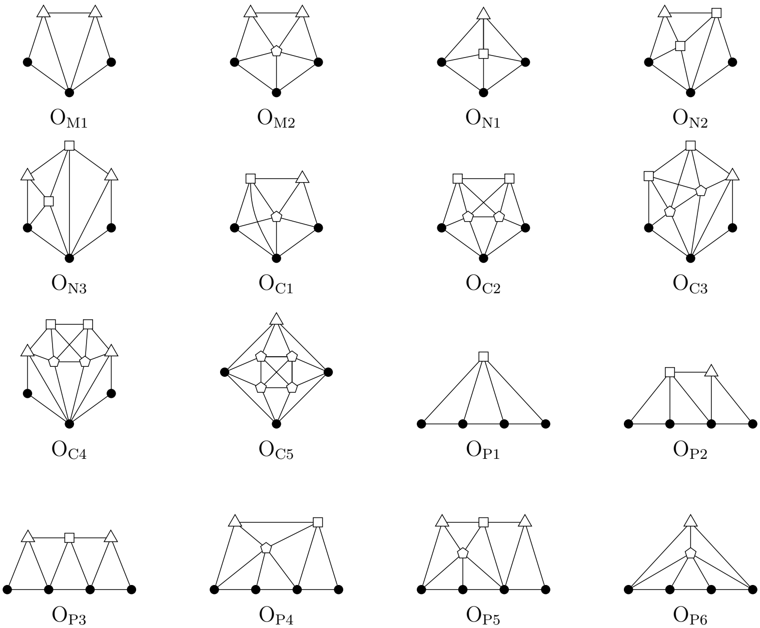


Figure 3: The obstructions used in Theorem 7

subgraphs in G' would still hold, and the induction hypothesis can be applied.

A subgraph $O \subseteq G$ is an *obstruction* if O is isomorphic to one of the graphs drawn in Figure 3 and sizes of the lists of its vertices match those prescribed by the figure, where the full-circle vertices have list of size one, triangle vertices have list of size three, square vertices have list of size four and pentagonal vertices have list of size five. Let us remark that if the distance condition holds, then G can contain at most one of the obstructions. For further reference we exhibit in Figure 4 all possible list assignments for which the obstructions are not colorable. In particular, observe that the following holds:

(2) *Let H be one of the obstructions and let Q be the path in H consisting of full-circle vertices. Suppose that Q has length two and that H is neither* O_{M1} *nor* O_{C1}. *Let q be the middle vertex of Q and let L be a list assignment such that each vertex v drawn by a k-gon has $|L(v)| = k$, while the vertices of Q have lists consisting of all possible colors. Then there exists a color b such that every L-coloring ψ of Q with $\psi(q) \neq b$ extends to an L-coloring of H.*

We prove the following claim, which obviously implies our main result, Theorem 2.

Theorem 7. *Let G be a graph drawn in the plane, P a path of length at most three contained in the boundary of the outer face F of $\mathcal{G}$ and L a list assignment for G. Suppose that there exist sets $N \subseteq V(G)$ and $M \subseteq E(G)$ such that L is (P, N, M)-valid and the drawing of G is (P, N, M)-distant. If*

(O) *every obstruction in G is L-colorable,*

then G is L-colorable.

Before giving the proof of this statement, let us give a quick outline. Essentially, we follow the proof of Theorem 6. First, we show that the outer face of a hypothetical minimal counterexample G has no chords and then we also restrict its 2-chords. This is somewhat

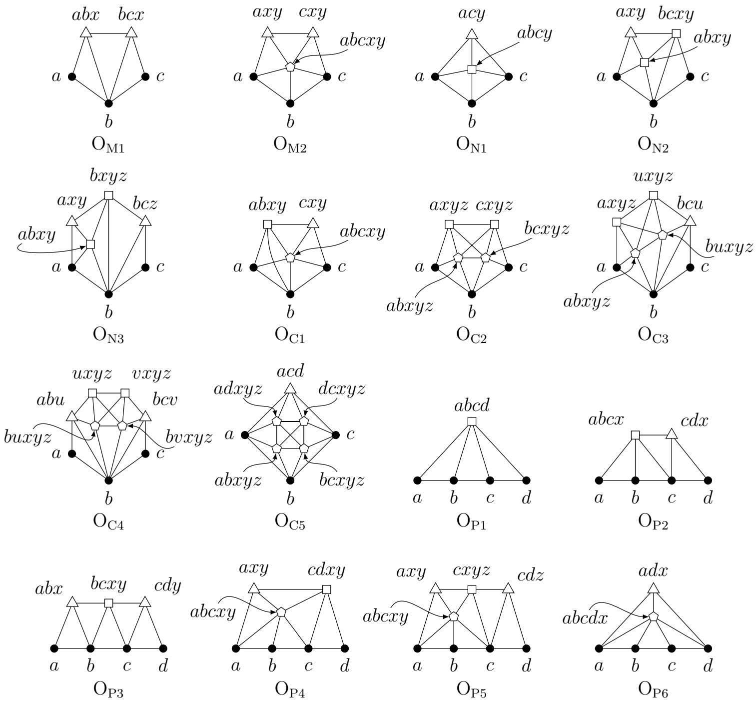
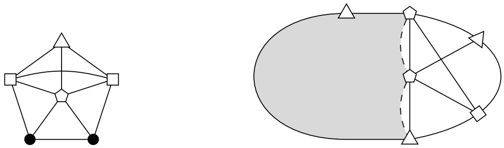


Figure 4: The lists for which the obstructions cannot be colored. Colors represented by different letters may be equal to each other if they do not occur in the same list for a particular obstruction.

more complicated due to the presence of crossings and the condition (O). Next, we find the set X and its partial coloring φ defined in the same way as in the proof of Theorem 6, and use it to construct the graph G' with the list assignment L'. By the minimality of G, we conclude that G' violates one of the assumptions of the theorem. A straightforward case analysis shows that (O) holds, and the conditions (S), (P) and (T) follow in the same way as in the proof of Theorem 6; but (M), (N) and (C) can be violated in ways which do not enable us to obtain a contradiction directly. However, we observe that in such a case, there is a special subgraph S near to X. In this situation, we apply the symmetric argument on the other side of the path P, and obtain another set X' and a special subgraph S' close to it. By the distance condition, we have $S = S'$, and thus there exists a short path from X to X' passing through S. In this situation, we consider all the possible combinations of X, X' and their positions relatively to S, and obtain a contradiction similarly to the way we deal with 2-chords.

Let us note that the assumption (C) is a product of a somewhat delicate tradeoff. We believe the claim still essentially holds even without this assumption, and avoiding it would greatly reduce the number of possible bad cases and simplify the last part of the proof. However, the list of obstructions in (O) would be significantly larger, making the first part of the proof longer and more complicated. Moreover, if we omit (C) completely, then there exists an obstruction with a precolored path of length one (see Figure 5(a)), which would be a major problem (we could not easily get rid of chords of F). One could consider excluding Figure 5(a) by forbidding vertices with lists of sizes three or four joined by a crossed edge. This would still simplify the last part of the proof a lot. However, in addition to having more than 10 new obstructions, we do not see a way how to reduce the 2-chord depicted in Figure 5(b), which would need to be dealt with somehow.

Proof of Theorem 7. We follow the outline of the proof of Theorem 6. We assume that G is a counterexample to Theorem 7 with $|V(G)| +$



(a) (b)

Figure 5: Why is condition (C) needed?

$|E(G)|$ as small as possible, and subject to that, with the minimum sizes of its lists. Let $k = \ell(P) \leq 3$, and let $P = p_0 p_1 \dots p_k$. By applying the same kind of reductions as used in the proof of Theorem 6 together with the minimality of G, one can show:

(3) *The following properties are satisfied:*

(a) *Every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.*

(b) *G is 2-connected and $\ell(P) \geq 1$.*

(c) *Every non-crossed chord of F is incident with exactly one internal vertex of P.*

(d) *If K is a triangle in G and no edge of K is crossed, then K is not separating. If K is a separating 4-cycle without crossed edges, then $\mathrm{Int}_K(G) - V(K)$ is either a vertex in N or a complete graph on 4 vertices involving a crossing.*

(e) *Every vertex $v \in V(G)$ satisfies $|L(v)| \leq 5$.*

(f) *If $v \in V(G) \setminus V(P)$ is adjacent to a vertex $p \in V(P)$, then $L(p) \subseteq L(v)$.*

Most properties in (3) are easy to argue about; they are left to the reader. Property (e) is achieved by removing colors from lists of size 6 or more. The only problem that may arise is that we obtain an obstruction; however, inspection of bad lists for the obstructions exhibited in Figure 4 shows that we can always remove one of the colors so that (O) still holds. The only remaining nontrivial claim is the property of separating 4-cycles in part (d). To prove that, we first color the subgraph of G consisting of $\mathrm{Ext}_K(G)$ and all chords of K and then consider $G' = \mathrm{Int}_K(G)$. Let $K = v_1v_2v_3v_4$, and let c_i be the color of v_i as used in the coloring of $\mathrm{Ext}_K(G)$. Suppose that $c_1 \neq c_3$. In that case we consider the list assignment L' for G' given by setting $L'(v) = L(v)$ if $v \notin V(K)$, $L'(v_i) = \{c_i\}$ for $i = 1, 2, 3$, and $L'(v_4) = \{c_1, c_3, c_4\}$. Since any L'-coloring of G' yields an L-coloring of G, we conclude that G' does not satisfy all assumptions of the theorem. It is easy to see that the only possible violation is that G' contains an obstruction. Note that this obstruction contains the whole path $v_1v_2v_3$ and that the only vertices whose lists have size 3 or 4 are v_4 and possibly a vertex in N. If a vertex in N is present, there is no crossing by the distance condition. The only obstructions with these properties are $\mathrm{O_{N1}}$ and $\mathrm{O_{C5}}$, yielding the outcome of the claim (a similar argument shows that $V(\mathrm{Int}_K(G))$ only consists of the vertices of the obstruction; see (4) below for details). The remaining case to consider is when $c_1 = c_3$. In this case we replace the color c_3 in the list of v_3 and in the lists of all its neighbors by a new color c_3' that does not occur elsewhere, and then apply the same argument as in the previous case. It is to be observed that the color c_3' will only be used for v_3, and the color c_3 will not be used on any of the neighbors of v_3. Thus a coloring with the revised lists gives rise to an L-coloring of G also in this case. This completes the proof of (3).

Let $T = v_1v_2v_3$ be a triangle in G. Suppose that the edge v_1v_2 is crossed by an edge uw, where w belongs to $G_2 = \mathrm{Int}_T(G)$ and $w \neq v_3$. Let φ be an L-coloring of $\mathrm{Ext}_T(G)$ and let L_2 be the list assignment such that $L_2(v_i) = \{\varphi(v_i)\}$ for $1 \leq i \leq 3$, $L_2(w) = L(w) \setminus \{\varphi(u)\}$ and $L_2(z) = L(z)$ otherwise. Note that $|L_2(w)| \geq 4$, that G_2 is not L_2-colorable and that it is $(v_1v_2v_3, (N \cap V(G_2)) \cup \{w\}, \emptyset)$-distant. Observe that G_2 satisfies the validity conditions (S)–(C), and also satisfies (O). Hence it is a counterexample to Theorem 7, contradicting the minimality of G. Similarly, if $w = v_3$, then we conclude that no vertex is drawn in the open disc bounded by T. Together with (3), we obtain the following conclusion:

(4) *If T is a triangle in G distinct from F, then $V(\mathrm{Int}_T(G)) = V(T)$.*

Suppose now that G contains one of the obstructions from Figure 3. Note that each of the obstructions contains a special subgraph. By the distance condition, none of them has further crossed edges and (4) implies that no such obstruction H appears in G, as otherwise we would have $G = H$ and G would be L-colorable by the assumptions.

Furthermore, analogous arguments as used in the proof of (3)(d) show that the following conditions hold:

(5) *If K is a 4-cycle in G distinct from F and $V(\mathrm{Int}_K(G)) \neq V(K)$, then either $\mathrm{Int}_K(G) - V(K)$ is K_4, or there exists a vertex z such that $V(\mathrm{Int}_K(G)) = V(K) \cup \{z\}$, z is adjacent to all vertices of K and z either belongs to N or is incident with an edge crossing an edge of K.*

(6) *If K is a 5-cycle in G distinct from F, $V(\mathrm{Int}_K(G)) \neq V(K)$, no edge of K is crossed and there exists a special subgraph $S \subseteq \mathrm{Ext}_K(G)$ such that $d(S, K) \leq 1$, then $V(\mathrm{Int}_K(G)) = V(K) \cup \{z\}$ for a vertex z adjacent to all vertices of K.*

Some explanation concerning the proof of (6) is needed: Again, we first color $\mathrm{Ext}_K(G)$ and then consider $\mathrm{Int}_K(G)$ with the 5-cycle precolored. By the previous results, K has no chords, since the outcomes of (5) would yield a special subgraph too close to S. Let uv be an edge of K, and let $G' = \mathrm{Int}_K(G) - \{u, v\}$. By removing the colors of u and v from the lists of their neighbors, we obtain another instance of a list coloring problem with a precolored path of length 2. Since any coloring of G' gives rise to a coloring of G, we conclude that one of the assumptions of the theorem is violated. By the distance condition, the only one that may not hold is the assumption (T). Since the common neighbor w of the three vertices on the path has list of size 3 (and it had list of size 5 in G), it is adjacent to u and v in G, thus proving the claim.

Our next goal is to show that F does not have chords. Let uv be a non-crossed chord of F. By (3)(c), u is an internal vertex of P, say $u = p_1$, while $v \not\in V(P)$. Let G_1 and G_2 be the uv-components for G such that $p_0 \in V(G_1)$, and let $P_1 = p_0p_1v$ and $P_2 = vp_1 \dots p_k$. For each color $c \in L(v) \setminus L(u)$, let L_c be the list assignment such that $L_c(v) = \{c\}$ and $L_c(z) = L(z)$ if $z \neq v$. Since G is not L-colorable, either G_1 or G_2 is not L_c-colorable. Furthermore, since both G_1 and G_2 are L-colorable (by the minimality of G), there exist distinct colors c_1 and c_2 such that G_1 is not L_{c_1}-colorable and G_2 is not L_{c_2}-colorable. Since G is a minimal counterexample, the assumptions of our theorem fail for G_1 and G_2 with respect to these list assignments. In the sequel we discuss what can go wrong.

All special subgraphs in G that do not contain v remain special in G_1 or G_2 and no new special subgraphs arise. Thus, G_1 is $(P_1, N \cap V(G_1), M \cap E(G_1))$-distant and G_2 is $(P_2, N \cap V(G_2), M \cap E(G_2))$-distant. Clearly, validity conditions (S), (N), (M), and (C) hold for both graphs. Thus one of (P), (T), or (O) fails. If G_1 contains an obstruction, then it contains a special subgraph whose distance to p_1 is at most two. In that case, we conclude that $\ell(P) \leq 2$ and that G_2 contains no obstruction, since the distance between special subgraphs in G is more than four; also, no edge at distance at most two from p_1 in G_2 is crossed. Since $\ell(P) \leq 2$, we may in this case exchange the role of G_1 and G_2 and henceforth assume that G_1 contains no obstructions. Similarly, by exchanging the roles of G_1 and G_2 if necessary, we may assume that no edge in G_1 at distance at most 2 from p_1 is crossed. Furthermore, if G_1 violates (T), then since no edge in G_1 incident with p_1 is crossed, we could consider the chord p_1z instead of p_1v, where z is the common neighbor of p_0, p_1 and v. Therefore, we can assume that G_1 satisfies (T) and (O). Since no L-coloring of G_2 extends to an L-coloring of G_1, we conclude that G_1 violates (P), and thus v is adjacent to p_0. Since vp_0 is neither crossed nor incident with an internal vertex of P, we conclude that vp_0 is part of the boundary of F, and hence G_1 is equal to the triangle p_0p_1v.

Suppose now that G_2 contains an obstruction H; by (3) and (4), we have $G_2 = H$. However, the inspection of the obstructions shows that G would either be L-colorable or an obstruction. Therefore, G_2 satisfies (O). Furthermore, by the absence of O_{P1} and property (T) of G, we conclude that there exists a color $c \in L(v) \setminus (L(p_0) \cup L(p_1))$ such that G_2 satisfies (P). Since this coloring does not extend to an L-coloring of G_2, it follows that G_2 violates (T), i.e., there exists a vertex w adjacent to v and to vertices $p, p' \in V(P) \setminus \{p_0\}$ such that $L(w) = L(p) \cup L(p') \cup \{c\}$. Since we cannot choose c so that G_2 satisfies both (P) and (T), it follows that either G contains O_{P2}, or $vw \in M$ (in which case $\ell(P) = 2$), and G contains O_{M1}. This is a contradiction, thus every chord of F is crossed.

Consider now a (crossed) chord uv of F that is not incident with an internal vertex of P. Let e be the edge crossing uv and let G_1 and G_2 be the uv-components of $G - e$ such that $P \subseteq G_1$. Let $e = x_1x_2$, where $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. By the minimality of G, there exists an L-coloring φ of G_1. Since $\varphi(u) \neq \varphi(v)$, we can assume that $\varphi(x_1) \neq \varphi(u)$. Let G' be the graph obtained from $G_2 - uv$ by adding new vertices y_1 and y_2, edges of the path $P' = uy_1y_2v$ and the edge y_1x_2. Let L' be the list assignment for G' such that $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(y_1) = \{\varphi(x_1)\}$, $L'(y_2) = \{c\}$ for a new color c that does not appear in any of the lists and $L'(z) = L(z)$ for any other vertex z. Note that G' has a new special subgraph consisting of the edge y_1y_2 and that G' is $(P', N \cap V(G'), M \cap E(G'))$-distant, since the crossing of G incident with x_2 does not belong to G' and any path from a special subgraph in G' to y_1y_2 passes through one of the vertices u, v, x_2 of the crossing in G. Furthermore, G' is not L'-colorable, and by the minimality of G, it violates (T) or (O). The latter is not possible, since y_2 has degree two, thus (T) does not hold in G'. This implies that x_2 has list of size three and it is adjacent to u and v. By (3) and (4), we have $V(G_2) = \{u, v, x_2\}$. Note that by (C), we conclude that each of $|L(u)|, |L(v)|, |L(x_1)|$ is either 1 or 5. Let a be a color in $L(x_2)$ distinct from the colors of its neighbors in P, which exists by (T). Let $G'' = G - x_2$ with the list assignment L'' such that $L''(z) = L(z) \setminus \{a\}$ for $z \in \{u, v, x_1\}$ and

$L''(z) = L(z)$ otherwise. Note that G'' is $(P, N \cup \{x_1\}, M\}$-distant and L'' is $(P, N \cup \{x_1\}, M)$-valid. By the minimality of G, we have that G'' violates (O). The obstruction involved is L-colorable, so it must contain one of the vertices whose lists have been changed. Since none of these vertices has list of size 3 or 4 in L and all special subgraphs are far away from the crossing, we conclude that $\ell(P) = 2$, $|L''(x_1)| = 4$ and G'' contains one of O_{N1}, O_{N2} or O_{N3}, in which the interior vertex with list of size 4 is x_1. However, inspection of these graphs shows that $|L''(u)| = 3$ or $|L''(v)| = 3$, which is a contradiction.

Finally, consider a crossed chord uv of F incident with an internal vertex u of P. Since G is (P, N, M)-distant, we have $\ell(P) = 2$, thus $u = p_1$. Let e be the edge crossing uv and let G_1 and G_2 be the uv-components of $G - e$ such that $p_0 \in V(G_1)$ and $p_2 \in V(G_2)$. Let $P_1 = p_0p_1v$ and $P_2 = p_2p_1v$, and let $e = x_1x_2$, where $x_i \in V(G_i)$ for $i \in \{1, 2\}$. Note that G_i is $(P_i, (N \cap V(G_i)) \cup \{x_i\}, M \cap E(G_i))$-distant. If G_i contains an edge f different from p_0p_1, p_0v, p_1v, then by the minimality of G there exists an L-coloring φ_{3-i} of $G - f \supseteq G_{3-i} + x_1x_2$. If additionally $|L(x_i)| \in \{1, 5\}$, then define L_i to be the list assignment for G_i such that $L_i(v) = \{\varphi_{3-i}(v)\}$, $L_i(x_i) = L(x_i) \setminus \{\varphi_{3-i}(x_{3-i})\}$, and $L_i(z) = L(z)$ for any other vertex z. Observe that G_i is not L_i-colorable, and we conclude that it violates (P), (T) or (O). (For (S) to hold, we add x_i to N). Since φ_{3-i} is a coloring of $G - f$, (P) is satisfied for G_i. Since G is (P, N, M)-distant and contains no non-crossed chord, it follows that G_i satisfies (T). Thus, G_i violates (O). The corresponding obstruction is O_{N1} since all others either have a special subgraph of G that would violate the distance condition in G, or have a non-crossed chord incident with p_1. Together with (3) and (4), we have that for each $i \in \{1, 2\}$, one of the following holds:

- $x_i \in V(P_i)$ and either $G_i = P_i$ or G_i is the triangle on $V(P_i)$, or
- $|L(x_i)| \in \{3, 4\}$, or

- G_i is equal to O_{N1} and x_i is its vertex with list of size four.

Since we already excluded all chords not incident with p_1, at most one of x_1 and x_2 has list of size three or four. By symmetry assume that $|L(x_1)| \in \{1,5\}$. If $|L(x_2)| \in \{1,5\}$, then all the possible combinations of such graphs G_1 and G_2 are either L-colorable or equal to O_{C1}. Therefore, $|L(x_2)| \in \{3,4\}$. Since every chord of F is crossed and incident with an internal vertex of P, we have $x_1 \notin V(F)$, thus G_1 is O_{N1}. Let w be the vertex of G_1 with list of size three, $G' = G - \{w, p_0\} - p_1v$ and L' the list assignment such that $L'(x_1) = \{\varphi_1(x_1)\}$, $L'(v) = \{\varphi_1(v)\}$ and $L'(z) = L(z)$ otherwise. Note that G' is $(p_2p_1x_1v, N \cap V(G'), M \cap E(G'))$-distant and not L'-colorable. If v has degree at least 5 in G, then it has degree at least three in G'. Together with (3), this implies that x_2 is not adjacent to v, hence (T) holds. If v has degree at most four, then $|L(v)| \leq 4$, and by (C), $|L(x_2)| = 4$, and again (T) holds. Therefore, G' violates (O). Since x_1 has degree three in G' and it is adjacent to a vertex with list of size three or four, G' contains (and by (3), is equal to) O_{P1} or O_{P2}. However, then G is L-colorable. Therefore, we obtain the following conclusion:

(7) *F has no chords.*

An easy corollary is that

(8) *no vertex of P is incident with a crossed edge.*

Indeed, if vp were a crossed edge with $p \in V(P)$, then by (7), we have $v \notin V(F)$. Furthermore, since P is incident with a crossing, we have $\ell(P) \leq 2$. Let L' be the list assignment such that $L'(v) = L(v) \setminus L(p)$ and L' matches L on the rest of the vertices of G. Note that $G - vp$ is not L'-colorable, and by the minimality of G, we conclude that $G - vp$ contains O_{N1}, O_{N2} or O_{N3}, whose internal vertex with list of size 4 is v. It cannot contain O_{N1}, since v is not adjacent to all vertices of P in $G - vp$. Similarly, it cannot contain O_{N3}, since the edge vp would be crossed twice. If $G - vp$ contains O_{N2}, then G contains O_{C1}. Comparison of bad lists for O_{N2} and O_{C1} in

Figure 4 shows that $\mathrm{O_{C1}}$ is not L-colorable, which is a contradiction to the assumption that (O) holds for G and L.

Consider now a vertex v with three neighbors $p_a, p_b, p_c \in V(P)$, where $a < b < c$. Let K be the cycle $p_a p_{a+1} \ldots p_c v$, and note that K has a chord vp_b. By (8), none of the edges vp_a, vp_b and vp_c is crossed. By (3)(d), K is not separating ((3)(d) allows a vertex of N or a K_4 with a crossed edge in the interior of K; however, this would only be possible if $\ell(P) = 3$, yielding two special subgraphs at distance 1). Suppose that $c - a = \ell(P)$, and let G_2 be the $p_a v p_c$-component of G that does not contain P. Since $v \notin V(F)$, and $v \notin N$ if $\ell(P) = 3$, there is a color in $L(v)$ that does not appear in the lists of vertices in P. Observe that G_2 (with the precolored path $p_a v p_c$) violates either (T) or (O). In the former case, G is either $\mathrm{O_{N1}}$ or $\mathrm{O_{P6}}$. In the latter case, we have $\ell(P) = 2$ by the distance condition, and (2) implies that G_2 is either $\mathrm{O_{M1}}$ or $\mathrm{O_{C1}}$. If G_2 is $\mathrm{O_{M1}}$, then G is $\mathrm{O_{M2}}$, and if G_2 is $\mathrm{O_{C1}}$, then G is L-colorable.

Finally, consider the case that $\ell(P) = 3$ and v is adjacent to say p_0, p_1 and p_2 and is not adjacent to p_3. If $L(p_0) = L(p_2)$, then $G - vp_2$ is a counterexample to Theorem 7 contradicting the minimality of G. Therefore, $L(p_0) \neq L(p_2)$. Since the edges vp_0, vp_1, and vp_2 are not crossed, the degree of p_1 is three. Let $G' = G - p_1 + p_0 p_2$, with the list assignment L' such that $L'(v) = L(v) \setminus L(p_1)$ and $L'(z) = L(z)$ otherwise. Note that G' is $(p_0 p_2 p_3, N \cup \{v\}, M)$-distant, since the rank of the special subgraph $p_1 p_2$ in G is greater than the rank of the special subgraph v (in G'), and any path Q between two special subgraphs S_1 and S_2 that uses the new edge $p_0 p_2$ gives rise to paths between S_1 or S_2 and the middle edge $p_1 p_2$ of P in G, thus implying $\ell(Q) \geq 14 + r(S_1) + r(S_2) + 2r(p_1 p_2) - 1 > 7 + r(S_1) + r(S_2)$. We conclude that G' violates (O) and contains $\mathrm{O_{N1}}$, $\mathrm{O_{N2}}$ or $\mathrm{O_{N3}}$ that is not L'-colorable; however, then G contains a non-L-colorable obstruction $\mathrm{O_{P6}}$, $\mathrm{O_{P4}}$ or $\mathrm{O_{P5}}$, respectively. Therefore, we have:

(9) *Every vertex has at most two neighbors in P.*

Suppose now that uv and xy are edges crossing each other and $u, x \in V(F)$. By (8), neither u nor x belongs to P. Let c be the curve formed by the part of the edge uv between u and the crossing and the part of the edge xy between the crossing and x. If c is not part of the boundary of F, then let G_2 be the subgraph of G drawn inside the closed disc bounded by c and the part of the boundary of F between u and x that does not contain P. Note that there are two possible situations, depending on whether G_2 includes the vertices v and y or not. In any case, we can write $G = G_1 \cup G_2$, where $G_1 \cap G_2$ consists only of vertices u and x. Let G_2' be the graph obtained from G_2 by adding a common neighbor w of u and x. No L-coloring of G_1 extends to an L-coloring of G_2' (where w is assigned an arbitrary color different from the colors of u and x). By the minimality of G, we conclude that G_2' (with the precolored path uwx) violates (P), thus $ux \in E(G)$. The conclusion is:

(10) *If u and x are crossing-adjacent and $u, x \in V(F)$, then either $ux \in E(G)$, or the crossing incident with u and x belongs to the boundary of F.*

Similarly, we derive the following property:

(11) *Suppose that $Q = x_1x_2 \dots x_{t-1}x_t$ is a path in G, where $t \leq 6$ and x_1x_2 crosses $x_{t-1}x_t$. Let c be the closed curve consisting of the path $x_2 \dots x_{t-1}$ and parts of the edges x_1x_2 and $x_{t-1}x_t$, and let X be the set of vertices of G drawn in the open disc bounded by c. If $x_1 \notin X$, then $X = \emptyset$.*

The proof of (11) proceeds as follows. Observe first that the curve c is not crossed since all its edges are close to a crossing. If the path $x_2 \dots x_{t-1}$ is induced in G, then the subgraph of G drawn inside the closed disc bounded by c, with the precolored path $x_2x_3 \dots x_{t-1}$, would be a counterexample to Theorem 7 smaller than G (the distance constraints are satisfied even if $t = 6$, since the middle edge of the path $x_2x_3x_4x_5$ has smaller rank then the crossing, whose distance to x_3x_4 in G is one). If $x_2 \dots x_{t-1}$ contains a chord x_ix_j ($i < j$), then we first consider $Q' = x_1x_2 \dots x_ix_j \dots x_t$ and then apply (3)(d) to show that no vertices are contained in the interior of the cycle $x_ix_{i+1} \dots x_jx_i$.

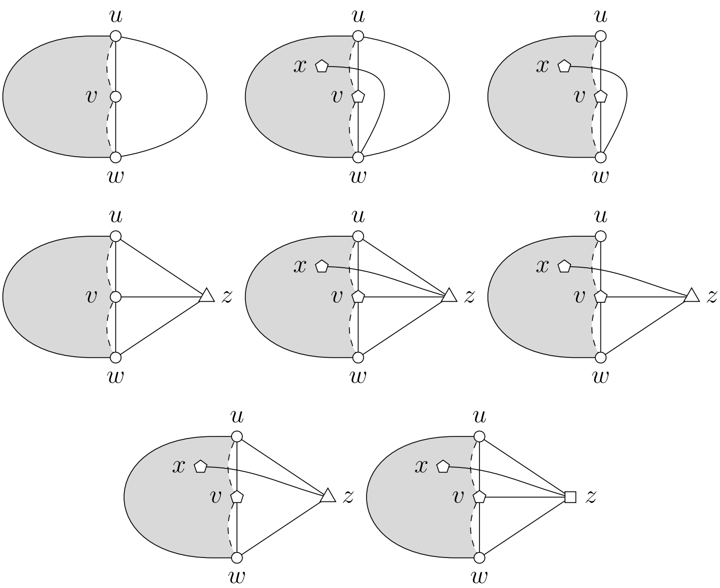


Figure 6: Possible cases for G_2 for a 2-chord uvw

Now, we shall consider the 2-chords of F.

(12) *Let uvw be a 2-chord of F such that vw is not crossed. Let c be the closed curve bounding the outer face of G and q the curve corresponding to the 2-chord uvw. Let c_1 and c_2 be the simple closed curves in $c \cup q$ distinct from c, and let G_1 and G_2 be the subgraphs of G drawn inside c_1 and c_2, respectively, so that $G_1 \cap G_2 = uvw$ and $G_1 \cup G_2$ is equal to G if the edge uv is not crossed, and is equal to $G - xy$ if uv is crossed by the edge xy. If neither u nor w is an internal vertex of P and $P \subseteq G_1$, then one of the following holds (cf. Figure 6):*

- *$V(G_2) = \{u, v, w\}$, and either uv is not crossed and $uw \in E(G)$, or uv is crossed by an edge incident with w; in the latter case, uw may or may not be an edge.*
- *$V(G_2) = \{u, v, w, z\}$ for a vertex z with list of size three, and either uv is not crossed and $uz, vz, wz \in E(G)$, or uv is crossed by an edge incident with z, $zw \in E(G)$ and at least one of uz and vz is an edge.*
- *$V(G_2) = \{u, v, w, z\}$ for a vertex z with list of size four adjacent to u, v, w and incident with an edge crossing uv.*

Proof. Let us consider a 2-chord uvw that does not satisfy the conclusion of the claim such that G_2 is maximal. First, suppose that uv is not crossed. An L-coloring of G_1 does not extend to an L-coloring of G_2, hence G_2 (with the precolored path uvw) violates (P), (T) or (O). If G_2 violates (P) or (T), then by (4) and (7) the outcome of the claim holds. Therefore, we conclude that G_2 violates (O). Since the obstruction in G_2 violating (O) contains a special subgraph with a vertex distinct from v and $v \notin V(F)$, we conclude that $v \notin N$, and hence $|L(v)| = 5$. By (3) and (4) we also conclude that G_2 is the obstruction. Let S be the set of L-colorings of uvw that do not extend to an L-coloring of G_2. The inspection of the non-colorable obstructions with $\ell(P) = 2$ in Figure 4 shows that one of the following holds:

(R1) there exists a set A of at most two colors and S contains only colorings ψ such that $\psi(v) \in A$, and furthermore, if $|A| = 2$ then neither u nor w has list of size three; or,

(R2) S contains only colorings ψ such that $\psi(u) = \psi(w)$, and neither u nor w has list of size three.

Indeed, by (2), all obstructions except for O_{M1} and O_{C1} satisfy (R1) with $|A| = 1$. If G_2 is O_{M1} or O_{C1}, then neither u nor w has list of size three, by (M) together with the distance condition and by (C). The inspection of the colorings shows that if G_2 is O_{C1}, then (R1) holds with $|A| = 2$, and if G_2 is O_{M1}, then either (R1) holds with $|A| = 2$, or (R2) holds (the latter is the case when the two lists of size 3 are equal, i.e., $a = c$ in Figure 4).

If (R1) holds, then let $G' = G_1$, with the list assignment L' such that $L'(v) = L(v) \setminus A$ and $L'(z) = L(z)$ for $z \neq w$. Note that if $|A| = 2$, then v has no neighbor in G_1 with list of size three by (R1) and by the maximality of G_2. If (R2) holds, let $G' = G_1 + uw$ with the list assignment $L' = L$. In either case, G' is not L'-colorable and it is $(P, N \cap V(G'), M \cap E(G'))$-distant (in the latter case, any path Q between special subgraphs H_1 and H_2 using the added edge uw gives rise to paths from H_1 and H_2 to the special subgraph of G_2, and thus $\ell(Q) \geq 14 + r(H_1) + r(H_2) - 3$). Furthermore, G' satisfies (T) by (7) and (9), and if G' violated (C) or (O), then v or uw would have to belong to a crossing or to an obstruction in G', and the distance between its special subgraph and the special subgraph of G_2 would be at most 4. Note that G' cannot violate (P), as otherwise $u, w \in V(P)$ and G_2 is O_{M1}, and by (3) and (9), v would have degree four and list of size five. Therefore, G' is a counterexample to Theorem 7 smaller than G, which is a contradiction.

Suppose now that uv is crossed by an edge xy, where $x \in V(G_1)$ and $y \in V(G_2)$. If $y = w$, then the conclusion of the claim holds by (10), hence assume that $y \neq w$. Furthermore, $x \neq w$ by (10), and $uw \notin E(G)$ by (4). Let G_1' be the graph obtained from G_1 by adding the edges ux and vx (if they are not present already). Note that this can be done without introducing any new crossings. Since u, v and x

are incident with a crossing in G, G_1' is $(P, N \cap V(G_1'), M \cap E(G_1'))$-distant. Furthermore, G_1' does not contain any obstruction, as its special subgraph would be at distance at most 2 from the crossing. By (7), u has at most one neighbor in P within G, hence G_1' satisfies (T). By (8), u and x cannot belong to P, hence by the minimality of G, there exists an L-coloring φ of G_1'. Let G_2' be the graph obtained from $G_2 - uv$ by adding the vertex x and edges ux, vx, yx. Consider the list assignment L_2' for G_2' such that $L_2'(z) = \{\varphi(z)\}$ for $z \in \{u, v, w, x\}$ and $L_2'(z) = L(z)$ otherwise. Note that G_2' is not L_2'-colorable and that it is $(uxvw, N \cap V(G_2'), M \cap E(G_2'))$-distant.

Since $y \neq w$ and since $uw \notin E(G_2')$, the graph G_2' satisfies (P) for the φ-colored path $uxvw$. If G_2' violates (T), then by (4) we have that $|L(y)| = 3$ and y is adjacent to at least two of u, v and w. In particular, $y \in V(F)$. Observe that if $vy \in E(G)$, then the yvw-component K that does not contain P can only be a triangle (since otherwise any L-coloring of the other yvw-component K' would extend to K, and K' would contradict the minimality of G). Thus $yw \in E(G)$. By (11) applied to the path $xywvu$, we have $V(G_2) = \{u, v, w, y\}$ and the conclusion of the claim holds. Let us now consider the remaining case that G_2' satisfies (T), and thus by the minimality of G, it violates (O). By (3) and (4), G_2' is equal to one of $\mathrm{O}_{\mathrm{P1}}, \ldots, \mathrm{O}_{\mathrm{P6}}$, but not O_{P3} since x has degree 3 in G_2'.

If G_2' is O_{P1}, then the conclusion of the claim holds. Otherwise, let us define S as the set of colorings ψ of the path $uxvw$ that do not extend to an L-coloring of G_2' and satisfy $\psi(u) \neq \psi(v)$. The inspection of the obstructions and their problematic list assignments displayed in Figure 4 shows that either (R1) or one of the following holds:

(R3) G_2' is O_{P2} and there exists a color c such that S contains only colorings ψ such that either $\psi(u) = c$ and $\psi(x) = \psi(w)$, or $\psi(x) = c$ and $\psi(u) = \psi(w)$. Moreover, $|L(u)| \neq 3$ and $|L(w)| \neq 3$.

(R4) G_2' is O_{P4} and there exists a color c such that S contains only colorings ψ satisfying either $\psi(v) = c$ or $\psi(x) = c$. Moreover,

$|L(u)| \neq 3$.

Again, the conclusions that the specified vertices do not have lists of size three follow in all applicable cases by noting that otherwise either (C) or the distance condition would be violated. Let us remark that for O_{P2} we have (R1) if the colors a, b, c, d in Figure 4 are different; we have (R3) if $b = d$ or $a = d$. To argue for O_{P4}, O_{P5}, O_{P6} we observe that $\psi(x)$ and $\psi(v)$ should be taken from the difference of the lists of the two neighbors of u (so these are colors b, c in Figure 4). This yields (R1) with the only exception in the case of O_{P4}, where we cannot argue about $|L(w)| \neq 3$, so we need (R4) in this (and only this) case.

The condition in (R3) that the lists of u and w do not have precisely three elements is argued as follows. Since x has degree 3 in G'_2, the vertex z of O_{P2} with list of size 3 is not the vertex y, and v, w are both adjacent to z. Since $|L(z)| = 3$ and the edge wz is close to a crossing in G, we conclude that $wz \notin M$ and hence $|L(w)| \neq 3$. Since $|L(y)| = 4$, (C) implies that $|L(u)| \neq 3$.

Now, the case when (R1) holds is handled in the same way as the case when uv was not crossed. If (R3) holds, then we let $G'_1 = G_1 + uw$ with the list assignment obtained from L by removing c from the list of u (note that $|L(u)| \neq 1$ by (8)); we may need to add an edge incident with u to M in order to satisfy (M). If (R4) holds and $|L(x)| = 5$, then let $G'_1 = G_1$ with the list assignment obtained by removing c from the lists of x and v (and adding x to N). In all the cases, G'_1 satisfies the assumptions of the theorem. Indeed, (P) is trivial, since $u \notin V(P)$ by (8). Similarly, (T) follows by (7) and (9). Finally, (O) holds since by the distance condition, we could only create O_{M1}, O_{M2}, O_{N1}, O_{N2} or O_{N3}, and each of them is excluded by (7) or (9). Therefore, G'_1 contradicts the minimality of G, since its coloring would extend to an L-coloring of G.

Finally, consider the case that (R4) holds and $|L(x)| \in \{3, 4\}$. By (10), all neighbors of u distinct from x belong to G_2. By (8), we have $u \notin V(P)$, $\deg(u) \geq |L(u)| \geq 4$, and thus u is adjacent to x and $|L(u)| = 4$. Since G'_2 is O_{P4}, every coloring of x, v and w extends to an L-coloring of G_2, hence G_1 contradicts the minimality of G. This completes the proof of (12). □

Similarly, one can prove the following:

(13) *Let $u, w \in V(F)$ be distinct vertices, neither of which is an internal vertex of P. Suppose that $v \notin V(F)$ is a vertex adjacent to w and crossing-adjacent to u. Let c be the closed curve not containing P consisting of vw, parts of the crossed edges incident with u and v, and a part of the boundary of F between u and w that does not contain P, and let G_2 be the subgraph of G drawn in the closed disc bounded by c. Then G_2 does not contain the crossing and satisfies one of the following:*

(a) $V(G_2) = \{u, v, w\}$ *and* $uw \in E(G)$, *or*

(b) $V(G_2) = \{u, v, w, z\}$, $|L(z)| = 3$ *and* z *is adjacent to* u, v *and* w.

Proof. By (12), it suffices to consider the case that $uv \notin V(G)$. Let G_1' be the graph obtained from G_1 as follows: If $uw \in E(G)$, then we add the edge uw. If u, v and w have a common neighbor z with list of size three, then we add z and incident edges. If $V(G_1') = V(G)$, then (a) or (b) holds. Otherwise, there exists an L-coloring ψ of G_1' by the minimality of G. Let L' be the list assignment such that $L'(v) = \{\psi(v)\}$, $L'(w) = \{\psi(w)\}$, $L'(u) = \{c\}$ for a new color c, $L'(x) = (L(x) \setminus \{\psi(u)\}) \cup \{c\}$ for each neighbor x of u distinct from v and w and $L'(x) = L(x)$ for all other vertices x. Note that $G_2' = G_2 + uv$ is not L'-colorable, and by the minimality of G, one of the assumptions of the theorem is violated in G_2'. By the construction of G_1' and the choice of ψ, (P) and (T) hold. By the distance condition, the only obstruction that can appear in G_2' is O_{C1}. However, then the 2-chord wvt (where t is the neighbor of u in G_2 with list of size four) contradicts (12). □

Let us now introduce a way of defining list assignments that will be used throughout the rest of the paper. Let φ be any proper partial L-coloring of G such that $\varphi(v) \notin L(p)$ for every pair of adjacent vertices $v \in \mathrm{dom}(\varphi)$ and $p \in V(P)$. For each vertex $z \in V(G)$, let

$$R_z = \bigcup_{p \in V(P) \setminus \mathrm{dom}(\varphi), zp \in E(G)} L(p).$$

We define L_φ to be the list assignment such that

$$L_\varphi(z) = \Big(L(z) \setminus \{\varphi(x) : x \in \mathrm{dom}(\varphi), xz \in E(G)\}\Big) \cup R_z.$$

Let us also define $G_\varphi = G - \mathrm{dom}(\varphi)$. Consider any L_φ-coloring ψ of G_φ. We claim that the combination of φ with ψ is a proper L-coloring of G. Indeed, for any $z \in V(G_\varphi)$, we clearly have $\psi(z) \notin R_z$, and thus $\psi(z) \in L_\varphi(z)$ is different from the colors of the neighbors of z in $\mathrm{dom}(\varphi)$. Since G is not L-colorable, we conclude that G_φ is not L_φ-colorable.

Suppose now that G contains a subgraph H isomorphic to one of the graphs drawn in Figure 3 such that the subgraph of H corresponding to full-circle vertices is equal to P, triangle vertices have lists of size *at least* three, square vertices have lists of size *at least* four and pentagonal vertices have lists of size five. Then we say that H is a *near-obstruction.*

(14) *If H is a near-obstruction, then H is isomorphic to one of* $\mathrm{O_{M1}}$*,* $\mathrm{O_{N2}}$*,* $\mathrm{O_{N3}}$ *or* $\mathrm{O_{P3}}$*. Furthermore,* $|(V(H) \cap V(F)) \setminus V(P)| \leq 1$*, and if* $(V(H) \cap V(F)) \setminus V(P) \neq \emptyset$*, then H is* $\mathrm{O_{N2}}$ *or* $\mathrm{O_{N3}}$*.*

Proof. By (9), H is isomorphic to one of $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$, $\mathrm{O_{C2}}$, $\mathrm{O_{C3}}$, $\mathrm{O_{C4}}$, $\mathrm{O_{C5}}$ or $\mathrm{O_{P3}}$.

By (12), if H is $\mathrm{O_{C5}}$, then $V(G) = V(H) \cup \{z\}$, where z is a vertex of degree three adjacent to p_0, p_2 and the vertex $w \notin V(P)$ in the outer face of H. However, the distance condition implies that $w \notin N$, so that $|L(w)| = 5$. This implies that G is L-colorable, which is a contradiction.

If H is $\mathrm{O}_{\mathrm{C}2}$, then let $p_0w_1w_2p_2$ be the path in the outer face of H. If $w_1, w_2 \in V(F)$, then $V(G) = V(H)$ and G is L-colorable by (O). Thus assume that $w_2 \notin V(F)$; hence $|L(w_2)| = 5$. If $w_1 \in V(F)$, then since w_2 has degree at least 5, by (12) we have that $V(G) = V(H) \cup \{z\}$, where z is adjacent to w_1, w_2 and p_2. However, then G is L-colorable. Therefore, $w_1 \notin V(F)$. Let φ be an L-coloring of H and G_2 the $p_0w_1w_2p_2$-component of G that does not contain P. Since φ does not extend to an L-coloring of G_2, it follows that G_2 with the precolored path $p_0w_1w_2p_2$ violates (P), (T) or (O). Since both w_1 and w_2 have degree at least 5, it follows that $p_0w_2 \notin E(G)$ and $w_1p_2 \notin E(G)$, hence (P) holds. Suppose that G_2 violates (T). Then a vertex z with list of size three is adjacent to three vertices among p_0, w_1, w_2 and p_2. If it is adjacent to all four of them, then G contains $\mathrm{O}_{\mathrm{C}5}$ which has already been excluded. Otherwise, since w_1 and w_2 have degree at least 5, by (5) z cannot be adjacent to p_0, w_1, and p_2. By symmetry, we may assume that z is adjacent to p_0, w_1, w_2. Then (12) applied to the 2-chord zw_2p_2 shows that there is a vertex z' adjacent to z whose list has size 3, and we see that the special edge zz' gives a contradiction. Finally, if G_2 violates (O), then the obstruction is equal to one of $\mathrm{O}_{\mathrm{P}1}$, $\mathrm{O}_{\mathrm{P}2}$, $\mathrm{O}_{\mathrm{P}3}$, $\mathrm{O}_{\mathrm{P}4}$, $\mathrm{O}_{\mathrm{P}5}$ or $\mathrm{O}_{\mathrm{P}6}$, and now it is easy to see (by comparing bad lists for the obstructions) that G is L-colorable.

If H is $\mathrm{O}_{\mathrm{C}3}$, then let w_1 be the vertex of H drawn by the triangle and w_2 the vertex of P that is not adjacent to it in G. If H is $\mathrm{O}_{\mathrm{C}4}$, then let w_1 and w_2 be the vertices of H drawn by triangles. By symmetry, we can assume that w_1 is the neighbor of p_2. Let $w_1x_1x_2w_2$ be the path in H formed by neighbors of p_1. Note that $|L(w_i)| \in \{1, 5\}$ by (7). Choose an L-coloring φ of the subgraph of G induced by $V(P) \cup \{w_1, w_2\}$ such that $\varphi(w_1) \neq \varphi(w_2)$ and either $|L_\varphi(x_1)| \geq 4$ or $L_\varphi(x_1) \neq L_\varphi(x_2)$. Note that this is possible since $|L(w_1)| = 5$. Let $G' = G - \{p_1, x_1, x_2\} + w_1w_2$ with the list assignment L' such that $L'(z) = \{\varphi(z)\}$ for $z \in \{w_1, w_2\}$ and $L'(z) = L(z)$ otherwise. Observe that G' is not L'-colorable (since every L'-coloring of G' extends to an L-coloring of G) and that it satisfies (P) for the precolored path $w_2w_1p_2$ or $p_0w_2w_1p_2$. By the minimality of

G, G' violates (T) or (O). In the former case, by symmetry we can assume that there exists a vertex $z \in V(G)$ such that $|L(z)| = 3$ and z is adjacent to p_2, w_1 and either w_2 or p_0. It follows that G contains a separating 4-cycle formed by non-crossed edges, and by (3) the interior of this 4-cycle contains K_4. By (7) and (12), there are no other vertices in G. Now, it is easy to see that the resulting graph is L-colorable. Therefore, G' violates (O). Since G is (P, N, M)-distant, this is only possible if H is $\mathrm{O_{C4}}$. In this case the obstruction in G' is one of $\mathrm{O_{P1}}$–$\mathrm{O_{P6}}$. Note that the edge w_1w_2 is contained in a triangle; let z be their common neighbor. By (3), the 4-cycle $w_1zw_2p_1$ surrounds K_4 in G. However, the graphs obtained from the obstructions $\mathrm{O_{P1}}$–$\mathrm{O_{P6}}$ with the precolored path $p_0w_2w_1p_2$ by adding the vertex p_1 joined to the vertices p_0, w_1, w_2, p_2, deleting the edge w_1w_2, and adding K_4 inside the 4-cycle $w_1zw_2p_1$ are easily seen to be L-colorable.

The remaining obstructions are $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$, and $\mathrm{O_{P3}}$. If H is $\mathrm{O_{M1}}$ or $\mathrm{O_{P3}}$, then none of the vertices in $V(H) \setminus V(P)$ belongs to F since this would contradict (7). In the other cases, at most one of the vertices of H can belong to F by the same reason. □

Observe that $\ell(P) \geq 2$, since if $\ell(P) < 2$, then we can precolor $2 - \ell(P)$ more vertices of F (by (8), we can extend P in the boundary of F). Let $p_k \ldots p_1p_0v_1v_2 \ldots v_s$ be the vertices contained in the boundary of F in the cyclic order around it. We either have $v_iv_{i+1} \in E(G)$, or v_i and v_{i+1} are crossing-adjacent, for each i. As we already observed, $p_0v_1, p_kv_s \in E(G)$. We also define $v_{s+1} = p_k$, $v_{s+2} = p_{k-1}, \ldots$.

If $s = 0$, then let φ be the L-coloring of p_0. Then G_φ with the list assignment L_φ is a smaller counterexample to Theorem 6. This contradiction shows that $s \geq 1$.

Suppose that $s = 1$ and let φ be the partial coloring that assigns a color in $L(v_1) \setminus (L(p_0) \cup L(p_k))$ to v_1. Note that if v_1 is adjacent to a vertex x by a crossed edge, then G_φ is $(P, N \cup \{x\}, \emptyset)$-distant, otherwise G_φ is $(P, N, \emptyset)$-distant. By the minimality of G, G_φ with the list assignment L_φ violates (O) and contains an obstruction H, which by (14) is one of O_{M1}, O_{N2}, O_{N3} or O_{P3}. Note that if $z \in V(H)$ is a vertex with list of size three according to L_φ, then z is adjacent to v_1 and belongs to N; but z is at distance at most one from the special subgraph in the obstruction, contradictory to the assumption that G is (P, N, M)-distant. Therefore, $s \geq 2$.

If v_1 is not adjacent to v_2 (i.e., v_1 and v_2 are crossing-adjacent), then let φ be a coloring that assigns a color from $L(v_1) \setminus L(p_0)$ to v_1 and the color from $L(p_0)$ to p_0. Note that $\ell(P) = 2$ by the distance condition. Let y be the vertex adjacent to v_1 by the crossed edge, and note that G_φ is $(p_1p_2, N \cup \{y\}, M)$-distant. By the minimality of G together with (3), (7) and (9), we conclude that G_φ with the list assignment L_φ violates (N) or (M). In the former case, we have $|L_\varphi(y)| = 3$ and since $|L(y)| = 5$, it follows that y is adjacent to p_0. However, by (4), v_2 would be adjacent to p_0, contrary to (7). In the latter case, p_0 and v_1 have a common neighbor $u \neq y$ adjacent to a vertex w with $|L(w)| = 3$. This contradicts (12). Therefore, $v_1v_2 \in E(G)$, and by symmetry, $v_{s-1}v_s \in E(G)$.

Suppose now that $s = 2$. By symmetry, assume that if v_2 is incident with a crossed edge, then v_1 is incident with a crossed edge as well. If $v_1v_2 \in M$, then let φ be an L-coloring of v_1 and v_2 such that $\varphi(v_1) \notin L(p_0)$ and $\varphi(v_2) \notin L(p_k)$. Otherwise, let φ be a coloring of v_1 by a color in $L(v_1) \setminus L(p_0)$ such that if $|L(v_2)| = 3$, then $\varphi(v_1) \notin L(v_2) \setminus L(p_k)$. Note that this is possible by (3)(f). Let us remark that when $|L(v_2) \setminus \{\varphi(v_1)\}| = 2$, then $L(p_k) = \{\varphi(v_1)\}$ and $L_\varphi(v_2) = L(v_2)$ by the definition of L_φ, and thus we always have $|L_\varphi(v_2)| \geq 3$. If G_φ with the list assignment L_φ violated (C), then v_2 would have to be incident with a crossing, and by the choice of v_1, the vertex v_1 would be incident with the same crossing, which then would not appear in G_φ. Therefore, G_φ satisfies (C). If v_1 is incident with a crossed edge v_1x, then let $N' = N \cup \{x\}$; if v_1 is adjacent to a vertex $y \in N$, then let $N' = N \setminus \{y\}$; otherwise let $N' = N$. If v_1 and v_2 have a common neighbor z belonging to N, then let $M' = M \cup \{v_2z\}$; otherwise let $M' = M \setminus \{v_1v_2\}$. Observe that G_φ is (P, N', M')-distant and that it satisfies (S), (N) and (M). By the minimality of G, G_φ violates (O) and thus G contains a near-obstruction H. By (14), H is O_{M1}, O_{N2}, O_{N3} or O_{P3}. Observe that $v_1v_2 \notin M$, since otherwise the distance between v_1v_2 and the special subgraph of H (which is also special in G) is at most 3. Every vertex with list of size three according to L_φ either belongs to N or is equal to v_2. If $v_2 \notin V(H)$, then H contains only one vertex with list of size three, hence H is O_{N2}. However, then N contains two adjacent vertices, which is a contradiction. Similarly, we exclude the case that $v_2 \in V(H)$ and H is O_{N3} or O_{P3}. Therefore $v_2 \in V(H)$ and H is O_{M1} or O_{N2}. The former is excluded by (7). If H is O_{N2}, then we have $V(G) = V(H) \cup \{v_1\}$ by (4). If v_1 is incident with a crossed edge, then G contains O_{C2}. On the other hand, if v_1 is not incident with a crossed edge, then $|L(v_1)| = 3$, $|L(v_2)| = 4$, $|N| = 1$ and G is L-colorable. This is a contradiction, hence $s \geq 3$.

Next, observe that if v_1 and v_2 are not crossing-adjacent, then $|L(v_1)| = 3$ or $|L(v_2)| = 3$. Otherwise, we could remove a color from the list of v_1. If the edges v_1x and e cross, then $|L(x)| = 5$ by (7), and both vertices incident with e have list of size five by (8) and (10), hence G with the new list assignment satisfies (C). By (14), no obstruction arises (since all vertices with lists of size three or four in the new list assignment are contained in $V(F)$). Thus G with the reduced lists satisfies (O) and contradicts the minimality assumption. Similarly, if v_1 and v_2 are crossing-adjacent and $|L(v_1)| > 3$ and $|L(v_2)| > 3$, then we can assume that $|L(v_1)| = |L(v_2)| = 4$.

If $|L(v_1)| = 3$ or $|L(v_2)| = 3$ and furthermore $v_1v_2, v_2v_3 \notin M$, then let the set $X \subseteq V(F) \setminus V(P)$ and its partial L-coloring φ be defined as in (X1)–(X4) in the proof of Theorem 6. Let us add two more cases for the situation when v_1 and v_2 are crossing-adjacent:

(X5) If $|L(v_1)| = |L(v_2)| = 4$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ is chosen arbitrarily.

(X6) If $|L(v_1)| = |L(v_2)| = 4$ and $|L(v_3)| = 3$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2) \setminus L(v_3)$ is chosen arbitrarily.

Let m be the largest index such that $v_m \in X$. Let us note that $m = 1$ in (X1) and (X5), $m = 3$ in (X4), and $m = 2$ otherwise. Also,

$X = \mathrm{dom}(\varphi)$ in all cases except for (X4b), when $X = \{v_1, v_2, v_3\}$ and $\mathrm{dom}(\varphi) = \{v_1, v_3\}$.

(15) *One of the following cases holds:*

(A1) $v_1v_2 \in M$ *or* $v_2v_3 \in M$.

(A2) *Either* v_1 *and* v_2 *or two distinct vertices in dom*(φ) *have a common neighbor in* N.

(A3) *There exists a crossing* q *and two crossing-adjacent vertices* $w_1, w_2 \in V(G_q)$ *such that* $V(G_q) \cap X = \emptyset$, w_1 *has a neighbor in dom*(φ) *and* w_2 *has two neighbors in dom*(φ).

(A4) $v_mv_{m+1} \in E(G)$, *there exists a crossing* q *such that* $V(G_q) \cap X = \emptyset$ *and* $v_{m+1}, v_{m+2} \in V(G_q)$, *and either* $|L(v_{m+1})| = |L(v_{m+2})| = 4$ *or* $|L(v_{m+1})| = 5$ *and* $|L(v_{m+2})| = 3$.

(A5) $v_mv_{m+1} \in E(G)$, $|L(v_{m+1})| \in \{3, 4\}$ *and there exists a crossing* q *such that* $V(G_q) \cap X = \emptyset$, $v_{m+1} \in V(G_q)$ *and a neighbor* $w \notin V(F)$ *of* v_m *is crossing-adjacent to* v_{m+1}.

(A6) $v_1 \notin X$ *and there exists a crossing* q *such that* $V(G_q) \cap X = \emptyset$, $v_1 \in V(G_q)$ *and a neighbor* $w \notin V(F)$ *of* v_2 *is crossing-adjacent to* v_1.

(A7) $|X| \geq 2$ *and there exists a path* $v_{m-1}xyv_{m+1}$, *where* x *and* y *are neighbors of* v_m *and* $y \in N$.

Proof. See Figure 7 for the illustration of the possibilities. Note that if (A1) does not hold, then X and φ are defined.

Suppose now for a contradiction that none of (A1)–(A7) holds. Let us consider the graph $G'_0 = G - \mathrm{dom}(\varphi)$ with the list assignment L_φ, and let G' be the graph obtained from G'_0 by repeatedly removing vertices whose list is larger than their degree. If $\mathrm{dom}(\varphi) \neq X$, then we have case (X4b) and $X \setminus \mathrm{dom}(\varphi) = \{v_2\}$. If v_2 is not incident with a crossing, then its degree in G'_0 is 1, and since $|L_\varphi(v_2)| \geq 2$, it is not

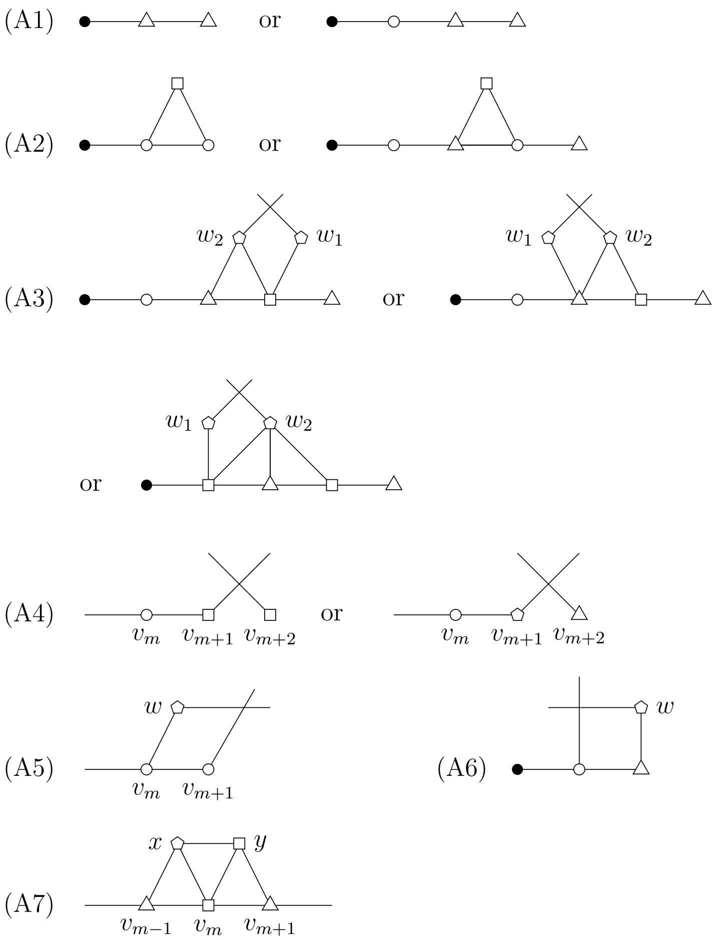


Figure 7: Possible outcomes of (15)

present in G'. On the other hand, if v_2 were incident with a crossing, then the fact that $|L(v_1)| = |L(v_3)| = 4$ would contradict (C). This shows that $G' \subseteq G - X$. Observe also that G' is not L_φ-colorable.

Next, we argue that G' satisfies the assumptions of Theorem 7 (with the sets $N' \subseteq V(G')$ and $M' \subseteq E(G')$ defined as the minimal sets such that (S), (N) and (M) hold), thus contradicting the minimality of G. The property (P) holds trivially, (T) holds by (9). To verify (S), note first that the only vertices not in the outer face of G' with list of size less than five are those belonging to N, or incident with a crossed edge joining them in G to a vertex in $\mathrm{dom}(\varphi)$; and the vertices of the latter kind have list of size four. Thus, they have been added into N' without violating the distance condition since the rank of special vertices in N' is smaller than the rank of the crossing. Next, suppose that a vertex $v \in V(G') \setminus V(P)$ satisfies $|L_\varphi(v)| \leq 2$. Note that $v \notin V(F)$ by the choice of X and φ and by (7). It follows that $v \in N$ and v has two colored neighbors in X, thus (A2) holds. This confirms that (S) is satisfied.

Now, let us consider property (C). Let q be a crossing in G' and suppose that (C) is violated at q, i.e., there exist distinct $u, v \in V(G_q)$ such that $|L_\varphi(u)| = 3$ and $|L_\varphi(v)| \in \{3, 4\}$. If both u and v belong to F, then by (7) and (10) we have that u and v are crossing-adjacent, $\{u, v\} = \{v_{m+1}, v_{m+2}\}$ and $L(v_{m+2}) = L_\varphi(v_{m+2})$. It follows that $|L(v_{m+1})| \neq 3$ and that (A4) holds. If $u \in V(F)$ and $v \notin V(F)$ and u and v are not crossing-adjacent, then since $V(G_q) \cap X = \emptyset$, (12) implies that (A4) holds. If $u \in V(F)$ and $v \notin V(F)$ and u and v are crossing-adjacent, then we apply (13). The outcome (a) of (13) gives (A5) or (A6). The outcome (b) gives a vertex $w \in X$ that is adjacent to v and a vertex z with $|L(z)| = 3$ that is adjacent to u, v and w. Therefore, $|L(u)| \neq 3$, so u has a neighbor in X. This is only possible in the subcase (X4a) of the definition of X, where $z = v_2$, $w = v_3$ and $u = v_1$, thus obtaining (A6). If $u \notin V(F)$ and $v \in V(F)$, then u has two neighbors in $\mathrm{dom}(\varphi)$. Since $V(G_q) \cap X = \emptyset$, (12) implies that u and v are crossing-adjacent. By (13), one of the neighbors of u in X is also adjacent to v and has list of size three, and by the choice of X, we conclude that (A6) holds. Finally, if $u, v \notin V(F)$, then they are crossing-adjacent by (4) and the fact that $V(G_q) \cap X = \emptyset$, and (A3) holds.

Therefore, we can assume that G' satisfies (C). Let us now consider the newly created special subgraphs in G'. If $v \in N' \setminus N$, then v is adjacent to a vertex of X by an edge containing a crossing q, and no other vertex of G_q belongs to X. Therefore, there is at most one such vertex. Consider now an edge $xy \in M' \setminus M$; we will show that either there exists a crossing q such that $\{x, y\} = V(G_q) \setminus X$, or at least one of x and y belongs to N. Note that xy has been added to M' because $|L_\varphi(x)| = |L_\varphi(y)| = 3$. Suppose that $x, y \notin N$. If $x, y \notin V(F)$, then both x and y have two neighbors in $\mathrm{dom}(\varphi)$. It is easy to see using (4) and (5) that this implies that x and y are crossing-adjacent in G via the edges joining x, y with X. If $x, y \in V(F)$, then by (7) we can assume that $x = v_{m+1}$ and $y = v_{m+2}$; but then $|L_\varphi(x)| \neq 3$ or $|L_\varphi(y)| \neq 3$ by the choice of X, which is a contradiction. Finally, suppose that say $x \in V(F)$ and $y \notin V(F)$; then y has two neighbors in $\mathrm{dom}(\varphi)$ and, in particular, we have cases (X2) or (X4). By (12), we have $x \in \{v_1, v_{m+1}\}$. If $x = v_1$, then y would be a common neighbor of v_1, v_2 and v_3, contradicting the choice of X (assumptions of (X4b) are satisfied, hence we would have $v_1 \in X$). If $x = v_{m+1}$, then $|L(v_m)| = 4$ and therefore one of the edges $v_{m-1}y$ and $v_{m+1}y$ is crossed since $\deg(v_m) \geq 4$. However, by the choice of X we have $|L(v_{m-1})| = |L(v_{m+1})| = 3$, contradicting (C).

It follows that $d(S_1, S_2) \geq 7 + r(S_1) + r(S_2)$ whenever S_1 is a special subgraph of G that is also special in G' and S_2 is any special subgraph of G'. Suppose now that S_1 and S_2 are both distinct newly created special subgraphs in G'. Note that $|N' \setminus N| \leq 1$ and if $N' \setminus N \neq \emptyset$, then $M' \setminus M = \emptyset$. It follows that $S_1, S_2 \in M' \setminus M$. As proved in the previous paragraph, each edge in $M' \setminus M$ is incident with a special subgraph in G that is adjacent to X. By the distance condition, we conclude that there exists a path xyz in G' such that $|L_\varphi(x)| = |L_\varphi(y)| = |L_\varphi(z)| = 3$ and $y \in N$. Note that at most one of x and z can have two neighbors in $\mathrm{dom}(\varphi)$, as otherwise G would contain a crossing at distance at most one from y; thus we may assume that $x \in V(F)$. By (12), $x \in \{v_1, v_{m+1}, v_{m+2}\}$. If $x = v_{m+2}$, then we would have $|L(v_{m+1})| = |L(x)| = 3$ and $v_{m+1}x \in M$ would be at distance one from $y \in N$, which is a contradiction; therefore, $x \neq v_{m+2}$. If $x = v_1$, then (A2) holds, hence $x = v_{m+1}$ and $z \notin V(F)$ has two neighbors in $\mathrm{dom}(\varphi)$. However, then $|L(v_m)| = 4$, hence $\deg(v_m) \geq 4$ and v_m is adjacent to y and (A7) holds. We conclude that G' is (P, N', M')-distant.

Finally, suppose that G' violates condition (O), and thus G contains a near-obstruction H. By (14), H is one of $\mathrm{O_{M1}}$, $\mathrm{O_{N2}}$, $\mathrm{O_{N3}}$ or $\mathrm{O_{P3}}$.

- If H is $\mathrm{O_{M1}}$, then let xy be the edge of H that belongs to M', where x is adjacent to p_2. Note that $x, y \notin V(F)$ by (7) and $xy \notin M$. If $x \notin N$, then x has two neighbors v_i and v_j in $\mathrm{dom}(\varphi)$, where $i < j$. By (12) applied to p_2xv_i, we have $j = i + 1$ and by the choice of X, $|L(v_j)| = 4$; hence v_j is incident with a crossing and thus $y \notin N$. Consequently, y is also adjacent to v_i and v_j. However, note that $|L(v_i)| = 3$, contradicting (C) for G. Therefore, $x \in N$ is adjacent to v_j, and y is adjacent to both v_i and v_j. By (12) applied to p_0yv_j, we have $i = 1$, $j = 2$ and $|L(v_1)| = 3$, and by (12) applied to p_2xv_2, we have that $s = 3$ and $|L(v_3)| = 3$. However, then G is L-colorable.

- Next, suppose that H is $\mathrm{O_{N2}}$ and let x and y be the vertices in the outer face of H such that $|L_\varphi(x)| = 3$ and $|L_\varphi(y)| = 4$. By (7), $y \notin V(F)$. If $x \in V(F)$, then by (12) we have $s \leq 2$, which is a contradiction, hence $x \notin V(F)$. Thus x has two neighbors in $\mathrm{dom}(\varphi)$ and y has one, and by (12) we conclude that $s = 3$ and $|L(v_1)| = |L(v_3)| = 3$. It follows that $X = \{v_1, v_2\}$, x is adjacent to v_1 and v_2, and y is adjacent to v_2. There are two cases, either v_2 is incident with a crossed edge or $|N| = 1$; in both of them, G is L-colorable.

- If H is $\mathrm{O_{N3}}$, then let xyz be the path in the outer face of H such that $|L_\varphi(x)| = |L_\varphi(z)| = 3$, $|L_\varphi(y)| = 4$ and z is adjacent to p_1. By (7), $z \notin V(F)$, thus z has two neighbors $w_1, w_2 \in \mathrm{dom}(\varphi)$, and by (12), we can assume that the neighbors of w_1 are w_2, z and an endvertex of P, and that $|L(w_1)| = 3$. Since $y \notin V(F)$, y is adjacent to w_2. Since x cannot have more than one neighbor in $\mathrm{dom}(\varphi)$, we have $x \in V(F)$. If $xw_2 \notin E(G)$, then (12) implies that x is adjacent to a vertex with list of size three, and thus $|L_\varphi(x)| = |L(x)| > 3$. This is a contradiction, hence $xw_2 \in E(G)$. By the choice of X, $|L(x)| = 3$. Again, we distinguish two cases depending on whether w_2 is incident with a crossed edge (in this case $|L(w_2)| = 5$ by (C)) or $|N| = 1$. In both cases, G is L-colorable.

- Therefore, H is $\mathrm{O_{P3}}$. But then two of the vertices of H have two neighbors in $\mathrm{dom}(\varphi)$, hence G contains a crossing at distance at most one from P, contradictory to the assumption that G is (P, N, M)-distant.

We have shown that G' satisfies all conditions of Theorem 7 for the list assignment L_φ. This gives a contradiction to the minimality of G and proves claim (15). □

Each case among (A1)–(A7) in (15) contains a special subgraph. Thus, G contains a special subgraph S whose distance from p_0 is at most $2 + r(S)$. Consequently, $\ell(P) = 2$. Next, we consider the set $X' \subseteq \{v_s, v_{s-1}, v_{s-2}\}$ defined symmetrically to X and conclude that there exists a special subgraph S' (satisfying one of (A1)–(A7) with v_i replaced by v_{s+1-i}) whose distance to p_2 is at most $2 + r(S')$. It follows that $d(S, S') \leq 6 + r(S) + r(S')$, and since G is (P, N, M)-distant, we have $S = S'$.

Next, we show that

(16) *S consists of two edges incident with a crossing.*

Proof. If not, then either $S \in M$ or $S \in N$. Suppose first that $S \in M$. Then (A1) holds and $s \leq 4$. Since $s \geq 3$, we can by symmetry assume that $S = v_2v_3$. If v_2, v_3 and v_i have no common neighbor for $i \in \{1, 4\}$ ($i = 1$ if $s = 3$), then let φ be an arbitrary

L-coloring of S (such that $\varphi(v_3) \notin L(p_2)$ if $s = 3$). Observe that G_φ cannot contain an obstruction since its special subgraph would be a special subgraph in G, too close to the special edge v_2v_3. Now it is easy to check using previously proved properties of G that G_φ satisfies all conditions of Theorem 7. (The same reasoning will be applied in the sequel without repeating it.) Therefore, G_φ with the list coloring L_φ is a counterexample to Theorem 7, contradicting the minimality of G. Thus, by symmetry, we may assume that v_1, v_2 and v_3 have a common neighbor w. In that case, w is not adjacent to v_4 by (12). Let φ be an L-coloring of v_1 and v_3 such that $\varphi(v_1) \notin L(p_0)$, $\varphi(v_3) \notin L(p_2)$ and $|L_\varphi(v_2)| \geq 2$. Then $G' = G - \{v_1, v_2, v_3\}$ with the list assignment L_φ contradicts the minimality of G since any L_φ-coloring of G' can be extended to v_2 by using a color in $L_\varphi(v_2)$, and can henceforth be extended to G.

Let us now consider the case that $S \in N$, hence (A2) or (A7) holds. Let i and j be the smallest and the largest integer, respectively, such that S is adjacent to v_i and v_j. By (12) we have $j \in \{i+1, i+2\}$. We consider the two possible values of j separately:

- Suppose first that $j = i + 1$. If $|X| \geq 2$, then $|L(v_m)| \geq 4$ and $|L(v_{m+1})| = 3$, hence (A7) cannot hold for both X and X'. If both X and X' satisfy (A2), then since $s \geq 3$, we can assume that $v_2, v_3 \in X$ have a common neighbor in N. By the choice of X, we have $|L(v_4)| = 3$, hence $s = 4$ and $v_2, v_3 \in X'$. However, then $|L(v_1)| \geq 4$ by the choice of X and $|L(v_1)| = 3$ by the choice of X', which is a contradiction.

  Hence, we can assume that (A7) holds for X and (A2) for X'; then we either have $s = m + 1$, or we have $s = m + 2$ and $X' = \{v_m, v_{m+1}\}$. If there exists an L-coloring φ of v_{m-1} and v_{m+1} such that their colors are distinct from the colors of their neighbors in P and $|L_\varphi(v_m)| \geq 3$, then $G' = G - \{v_{m-1}, v_m, v_{m+1}\}$ with the list assignment L_φ contradicts the minimality of G. (Observe that G' satisfies (O), since no special subgraph of G is at distance at most two from S. A new special subgraph would appear in G' only if S would be adjacent to v_{m+2}, which is not the case since $j = i + 1$.)

We conclude that no such coloring exists, hence both v_{m-1} and v_{m+1} have a neighbor in P and $s = 3$. Furthermore, $|L(v_1)| = 3$ and $L(v_1) \setminus L(p_0) \subset L(v_2)$. Let w' be the common neighbor of S and v_1. Suppose that there exists a color $c \in L(w')$ different from the colors of the neighbors of w' in P such that either $c \notin L(v_2)$, or v_1 has degree three and $c \notin L(v_1) \setminus L(p_0)$. In this case, we let φ be the partial coloring such that $\varphi(w') = c$ and let $G' = G - \{w', v_2\}$ if $c \notin L(v_2)$ and $G' = G - \{w', v_1, v_2\}$ if $c \in L(v_2)$. Observe that G' is not L_φ-colorable. Furthermore, it satisfies the assumptions of Theorem 7, with the edge Sv_3 belonging to M (the condition (O) holds by (14), the distance condition and (7)). This contradicts the minimality of G, and thus no such color c exists. Since $|L(w')| > |L(v_2)|$, it follows that w' has a neighbor in P. By (12), w' is not adjacent to p_2, hence it is adjacent to p_0 or p_1. However, then (5) and (7) imply that v_1 has degree three, and since $|L(v_1) \setminus L(p_0)| = 2$ and w' has at most two neighbors in P, the color c exists. This is a contradiction.

- It remains to consider the case when $j = i + 2$. In this case S is adjacent to v_i and v_{i+2}, and by (12) we conclude that v_{i+1} is a vertex of degree 3 with neighbors v_i, v_{i+2}, and S. Thus, $|L(v_{i+1})| = 3$. Suppose first that both X and X' satisfy (A7). If there exists a coloring φ of S by a color different from the colors of its neighbors in P such that $\varphi(S) \notin L(v_i) \cap L(v_{i+1}) \cap L(v_{i+2})$, then $G' = G - \{S, v_i, v_{i+1}, v_{i+2}\}$ with the list assignment L_φ is a counterexample contradicting the minimality of G (since in this case any L_φ-coloring of G' extends to an L-coloring of G). Otherwise, note that S is not adjacent to p_0 or p_2 by (12), hence S is adjacent to p_1 and $L(S) \setminus L(p_1) = L(v_{i+1}) \subseteq L(v_i) \cap L(v_{i+2})$. However, in this case we let φ be the L-coloring of X as chosen in the proof of Theorem 6, and note that $\varphi(v_i) \notin L(v_{i+1}) = L(S) \setminus L(p_1)$.

Then $G - X$ with the list assignment L_φ for any other vertex z is a counterexample contradicting the minimality of G.

Hence, we can assume that say X' satisfies (A2). Let us first consider the case that X satisfies (A2) as well. Note that $v_{i+2} \notin X$, as otherwise $|L(v_{i+3})| = 3$ by the choice of X, and thus $v_{i+1} \notin X'$, contradictory to the assumption that X' satisfies (A2). Symmetrically, $v_i \notin X'$. Since $|L(v_{i+1})| = 3$, we cannot have $\{v_i, v_{i+1}\} \subseteq X$, thus $i = 1$, and by symmetry, $s = 3$. Observe that we cannot color S by a color $\varphi(S) \notin L(v_{i+1})$, as otherwise $G - \{S, v_{i+1}\}$ with the list assignment L_φ would contradict the minimality of G. Therefore, S has a neighbor in P, and by (12), this neighbor is p_1. By (5), the 4-cycle $p_1p_0v_1S$ is not separating, and by (7), v_1 has degree three. This is a contradiction, since $|L(v_1)| > 3$.

Therefore, X satisfies (A7). Note that v_{i+1} cannot be the element of X' with the smallest index, thus $i + 2 = s$. As before, we exclude the case that S can be colored by a color not belonging to $L(v_i) \cap L(v_{i+1})$, hence S has a neighbor in P. By (12), S is not adjacent to p_0 or p_2, hence S is adjacent to p_1. However, by (5), the 4-cycle $p_1Sv_{i+2}p_2$ is not separating, and by (7), v_{i+2} is not adjacent to p_1. Thus, v_{i+2} has degree three and list of size at least four, which is a contradiction.

This completes the proof of the case when $S \in N$. □

Therefore, (16) holds and S consists of two edges incident with a crossing q. We conclude that each of X and X' satisfies one of (A3), (A4), (A5) or (A6). If one of them satisfies (A4), then $|V(G_q) \cap V(F)| = 2$ by (7). If it satisfies (A6), then by (7), (8) and (10) we have $|V(G_q) \cap V(F)| = 1$. If it satisfies (A3), then similarly $|V(G_q) \cap V(F)| \leq 1$, and if it satisfies (A5) then $1 \leq |V(G_q) \cap V(F)| \leq 2$.

(17) *Neither X nor X' satisfies* (A3).

Proof. Suppose for a contradiction that X satisfies (A3). Let w_1 and w_2 be as in the description of (A3). Note that w_2 is adjacent to v_{m-1} and v_m (even if $v_{m-1} \notin \mathrm{dom}(\varphi)$, in the case (X4b)) and that $|L(v_{m-1})| = |L(v_{m+1})| = 3$.

Let us first consider the case that $|V(G_q) \cap V(F)| = \emptyset$. In this case X' satisfies (A3) as well, i.e., there exists $w'_2 \in V(G_q)$ adjacent to v_b and v_{b+1}, where b is the smallest index of an element of X', and another vertex w'_1 of G_q that has one neighbor in X'. Since $|L(v_b)| \neq 3$, we have $b \notin \{m-1, m+1\}$. Consequently, $|X \cap X'| \leq 1$, and $w'_2 \neq w_2$ by (12).

We now distinguish two cases regarding whether w_2 is adjacent or crossing-adjacent to w'_2 in G_q.

- Suppose that $w_2 w'_2$ is a crossed edge. Then $b \neq m$ by (4) and the assumption that G_q is disjoint with F; thus $b \geq m+2$. Let G_1 and G_2 be the subgraphs of G intersecting in $v_m w_2 w'_2 v_b$, such that $G_1 \cup G_2 = G - e$, where e is the edge crossing $w_2 w'_2$, and $P \subset G_1$. By (12), we have that $w_1 \neq w'_2$, $w'_1 \neq w_2$ and that if $w_1 = w'_1$, then w_1 belongs to G_2. By symmetry, assume that w_1 belongs to G_2. If w_1 is adjacent to v_b, then $b = m+2$ by (12). Let $T = \{v_m, v_{m+1}, v_{m+2}, w_1\}$. By using (11) it is easy to see that $|L(t)| = \deg(t)$ for each $t \in T \setminus \{w_1\}$ and that $\deg(w_1) \leq 6$. By the minimality of G, there exists an L-coloring φ of $G - T$. Consider the subgraph G' of G induced by T with the list assignment L_φ. We have $|L_\varphi(v_{m+1})| \geq 3$ and $|L_\varphi(z)| \geq 2$ for $z \in T \setminus \{v_{m+1}\}$. If $L_\varphi(w_1) \neq L_\varphi(v_m)$, then we color w_1 by a color in $L_\varphi(w_1) \setminus L_\varphi(v_m)$ and extend this coloring to the rest of G'. Similarly, G' is L_φ-colorable if $L_\varphi(w_1) \neq L_\varphi(v_{m+2})$. If $L_\varphi(v_m) = L_\varphi(w_1) = L_\varphi(v_{m+2})$, then we color v_{m+1} by a color in $L_\varphi(v_{m+1}) \setminus L_\varphi(w_1)$ and again we can extend this to an L_φ-coloring of G'. It follows that G is L-colorable, which is a contradiction.

  Therefore, w_1 is not adjacent to v_b, and in particular $w_1 \neq w'_1$ and $w'_1 \in V(G_1)$. Let φ be an L-coloring of G_1, which exists by the minimality of G. Since w_1 is not adjacent to v_b, note

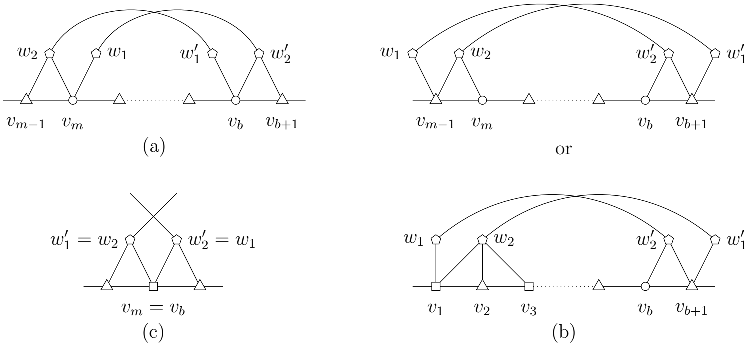


Figure 8: Subcases when w_2 and w_2' are crossing-adjacent

that w_1 has at most three neighbors in G_1 different from w_2'. Hence, we can additionally choose a color $\varphi(w_1)$ for w_1 different from the colors of its neighbors in G_1 so that $\varphi(w_1) \neq \varphi(w_2')$. Let $G_2' = G_2 - w_2 + w_1w_2'$. Note that G_2' gives an instance of Theorem 7 with the precolored path $P' = v_m w_1 w_2' v_b$, since the added edge w_1w_2' can be drawn without crossings following the crossed edges of G that are no longer in G_2'. It is clear that G_2' satisfies validity and distance constraints. Note that φ does not extend to an L-coloring of G_2'. Thus G_2' violates (T) or (O). In the former case, the vertex violating (T) must be v_{m+1} and we would have $b = m + 2$. Consequently, v_b would have degree at most three, which is a contradiction. In the latter case, since $|L(v_{m+1})| = |L(v_{b-1})| = 3$ and v_b has degree at least three in G_2', we have that G_2' is equal to O_{P5} or O_{P6}. In both cases, any L-coloring of $G_1 - \{v_m, v_b\}$ would extend to an L-coloring of G, a contradiction.

- Suppose now that w_2 is crossing-adjacent to w_2'. Let G_1 and G_2 be the subgraphs of G intersecting in $\{v_b, w_2', w_2, v_m\}$, where

$P \subset G_1$ and $G_1 \cup G_2$ is equal to G without the crossed edges. We have two subcases: either $b > m$ or $b = m$.

- If $b > m$, then (12) implies that w'_2 has no neighbor in X, and thus $w_1 \neq w'_2$. Symmetrically, $w'_1 \neq w_2$. Considering the drawing of G in the plane, we conclude that the edges of G_q are $w_1w'_2$ and w'_1w_2.

  If $w_1, w'_1 \notin V(G_1)$ (see Figure 8(a)), then $w_1v_m, w'_1v_b \in E(G)$. Let φ be an L-coloring of $G_1 + \{w_1w'_2, w'_1w_2, w_1w'_1\}$ which exists by the minimality of G, and note that φ does not extend to an L-coloring of $G'_2 = G_2 + w_1w'_1$. Observe that G'_2 provides an instance for Theorem 7 with the precolored path $v_mw_1w'_1v_b$. It is easy to see that we can choose the colors of w_1, w'_1, v_m and v_b so that G'_2 satisfies the assumptions of the theorem (once the coloring of $G_1 - \{v_m, v_b\}$ is fixed, we still have two possible choices for the colors of v_m and v_b and three possible choices for the colors of w_1 and w'_1). This is a contradiction. The case that $w_1, w'_1 \in V(G_1)$ (see Figure 8(b)) is excluded similarly.
- If $b = m$, then let w_2z and w'_2z' be the edges of G_q (note that we have $w_1 = w'_2$ and $w'_1 = w_2$). Suppose that $z, z' \in V(G_2)$. Note that $V(G_2) \neq \{z, z', w_2, w'_2, v_m\}$, since otherwise z would have degree at most four and $|L(z)| = 5$. Therefore, the subgraph of G induced by $V(G_1) \cup \{z, z'\}$ has an L-coloring ψ by the minimality of G. Let L' be the list assignment for $G'_2 = G_2 - \{z, z'\}$ obtained from L by removing the colors of z and z' according to ψ from the lists of their neighbors and by setting $L'(w_2) = \{\psi(w_2)\}$, $L'(v_m) = \{\psi(v_m)\}$ and $L'(w'_2) = \{\psi(w'_2)\}$. Note that G'_2 satisfies (O) by the distance condition and (P) by the choice of ψ, and since G is not L-colorable, we conclude that G'_2 violates (T). Therefore, G_2 contains a vertex adjacent to w_2, w'_2, v_m, z and z', and by (4), z and z' have degree at most four. This is a contradiction.

Therefore, we have $z, z' \in V(G_1)$ (see Figure 8(c)), and by (11), $\deg(v_m) = 4$. Let $S_1 = L(v_2)$ if $m = 3$ and $S_1 = L(v_1) \setminus L(p_0)$ if $m = 2$. Note that $S_1 \subset L(v_m)$, as otherwise we consider the partial coloring φ with $\varphi(v_{m-1}) \in S_1 \setminus L(v_m)$ and conclude that G_φ with the list assignment L_φ contradicts the minimality of G. Suppose that there exists a color $c \in L(w_2) \setminus L(v_m)$, or that $\deg(v_{m-1}) = 3$ and there exists a color $c \in L(w_2) \setminus S_1$, such that this color c is distinct from the colors of the neighbors of w_2 in P. Let $G' = G - \{w_2, v_m\}$ if $\deg(v_{m-1}) > 3$ and $G' = G - \{w_2, v_m, v_{m-1}\}$ if $\deg(v_{m-1}) = 3$, with the list assignment L' obtained from L by removing c from the lists of neighbors of w_2 and setting $L'(v_{m-1}) = L(v_{m-1})$ if v_{m-1} belongs to $V(G')$ (observe that $c \notin S_1$ and that in any L'-coloring of G', the color of v_{m-1} must belong to S_1). Note that L' is $(P, N \cup \{z\}, M)$-valid. Every L'-coloring of G' would extend to an L-coloring of G, thus G' is not L'-colorable. By the minimality of G, we conclude that G' violates (O), and by (14) and the distance condition, G' contains $\mathrm{O}_{\mathrm{N}2}$ or $\mathrm{O}_{\mathrm{N}3}$. However, then z is adjacent to two vertices of P and to z' and w_2', and at least one of z' and w_2' has a list of size three according to L', which is a contradiction since $|L(z')| = |L(w_2')| = 5$.

We conclude that there exists no such color c. Since $|L(v_m)| = 4$ and $|L(w_2)| = 5$, we conclude that w_2 has a neighbor in P. By (12), w_2 is not adjacent to p_2, and if it were adjacent to p_0, then we would have $m = 2$, $\deg(v_1) = 3$ and there would exist a color $c \in L(w_2) \setminus (S_1 \cup L(p_0) \cup L(p_1))$. Therefore, w_2 is adjacent to p_1. By symmetry, w_2' is adjacent to p_1 as well. However, the edges w_2p_1 and $w_2'p_1$ are not crossed by (8), and thus the crossing is contained inside the 4-cycle $v_mw_2p_1w_2'$, contrary to (5).

We conclude that $V(G_q) \cap V(F) \neq \emptyset$. By (7), $w_2 \notin V(F)$. Let

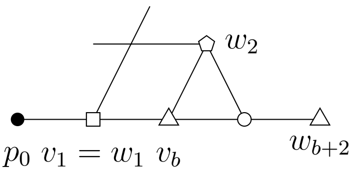


Figure 9: Subcase combining (A3) and (A5).

w be the vertex joined to w_2 by a crossed edge, and let w_1w' be the other crossing edge. Since $V(G_q) \cap X = \emptyset$, by (12) we have $w \notin V(F)$. Since v_m has degree at least four, we cannot have $w_1 = v_{m+1}$, thus by (7) and (8), we have $w_1 \notin V(F) \setminus \{v_1\}$. If $w_1 \notin V(F)$ and $x \in X$ is a neighbor of w_1, then the 2-chord xw_1w' separates P from either w_2 or w, and neither w_2 nor w belongs to F, contrary to (12). We conclude that $w_1 = v_1$ and $V(G_q) \cap V(F) = \{v_1\}$, hence $v_1 \notin X$ and X was chosen according to (X4a).

Since $|V(G_q) \cap V(F)| = 1$, X' must satisfy (A3), (A5) or (A6). If X' satisfied (A3), the conclusions of the preceding paragraph would apply symmetrically and we would have $v_1 = v_b$, which is a contradiction. Similarly, X' cannot satisfy (A6). The remaining possibility is that (A5) holds for X'. Then $v_1 = v_{b-1}$ and $v_b = v_2$. The situation is shown in Figure 9. Since X was chosen according to (X4a), we have $|L(v_b)| = |L(v_{b+2})| = 3$; in particular, $s \geq 4$ and $b \leq s - 2$. This is only possible if X' has been chosen according to (X4), but then $|L(v_b)| > 3$. This is a contradiction, showing that (A3) does not occur. □

Next, we claim that

(18) $|V(G_q) \cap V(F)| = 1$.

Proof. Since X does not satisfy (A3), if $|V(G_q) \cap V(F)| \neq 1$ then $|V(G_q) \cap V(F)| = 2$ and each of X and X' satisfies (A4) or (A5). By

(7) and (10), $V(G_q) \cap V(F) = \{v_{m+1}, v_{m+2}\}$ and v_{m+1} is crossing-adjacent to v_{m+2}. Let $v_{m+1}w$ and $v_{m+2}w'$ be the crossed edges. By symmetry, we can assume that $|L(v_{m+1})| \geq |L(v_{m+2})|$. By (C), either $|L(v_{m+1})| \geq |L(v_{m+2})| \geq 4$ or $|L(v_{m+1})| = 5$ and $|L(v_{m+2})| = 3$. Therefore, X was chosen according to the rules (X1) or (X3) and $|L(v_m)| = 3$.

If $L(v_{m+2}) \neq L(v_{m+1})$, then let c be a color in $L(v_{m+1}) \setminus L(v_{m+2})$. If v_{m+1} is not adjacent to v_{m+2}, then let c be an arbitrary color in $L(v_{m+1})$. In both cases, let φ be an L-coloring of v_m and v_{m+1} such that $\varphi(v_{m+1}) = c$ and if $m = 1$, then $\varphi(v_1) \notin L(p_0)$. It is easy to see that L_φ is a $(P, N \cup \{w\}, M)$-valid list assignment for G_φ. Therefore, G_φ violates (O). By (14), G_φ contains $\mathrm{O_{N2}}$ or $\mathrm{O_{N3}}$. It follows that w is adjacent to p_1 and to p_0 or p_2. However, if w is adjacent to p_0, then by (12), v_{m+2} is incident with a chord of F, contradicting (7). If w is adjacent to p_2, then v_{m+2} has degree at most three in G_φ by (12), and since $|L_\varphi(v_{m+2})| \geq 4$, G_φ contains neither $\mathrm{O_{N2}}$ nor $\mathrm{O_{N3}}$. This is a contradiction, implying that $L(v_{m+1}) = L(v_{m+2})$ (and in particular, $|L(v_{m+1})| = |L(v_{m+2})| = 4$), and $v_{m+1}v_{m+2} \in E(G)$. By the choice of X', we have $|L(v_{m+3})| = 3$.

Suppose now that $w'v_m \in E(G)$. Note that v_{m+1} has degree at least four, so it is adjacent to w'. Let $S_1 = L(v_m)$ if $m \neq 1$ and $S_1 = L(v_m) \setminus L(p_0)$ if $m = 1$. Note that $S_1 \subseteq L(v_{m+1})$, as otherwise we can choose an L-coloring φ of v_m such that $\varphi(v_m) \in S_1 \setminus L(v_{m+1})$, and $G_1 = G - \{v_m, v_{m+1}\}$ with the list assignment L_φ is a counterexample contradicting the minimality of G (note that G_1 cannot contain an obstruction since no internal vertex in G_1 has its list decreased and q is not a crossing in G_1). Since $L(v_{m+1}) = L(v_{m+2})$, we conclude that $S_1 \subseteq L(v_{m+2})$. Let G' be the graph obtained from $G - v_{m+1}$ by identifying v_m with v_{m+2}, and give the resulting vertex z the list of v_m. Note that G' satisfies the validity and the distance conditions of Theorem 7 (with the edge zv_{m+3} added to M). Since every coloring of G' gives rise to an L-coloring of G, condition (O) is violated in G' by the minimality of G. However, G' contains neither $\mathrm{O_{M1}}$ nor $\mathrm{O_{M2}}$ (and the exclusion of other obstructions is obvious). Therefore, $w'v_m \notin E(G)$, and by symmetry, $wv_{m+3} \notin E(G)$.

Let $S_2 = L(v_{m+3})$ if $m + 3 \neq s$ and $S_2 = L(v_{m+3}) \setminus L(p_2)$ if $m + 3 = s$. Suppose now that there exists an L-coloring φ of v_{m+1} and v_{m+2} such that $\varphi(v_{m+1}) \notin S_1$ and $\varphi(v_{m+2}) \notin S_2$. Then L_φ is a $(P, N, \{ww'\})$-valid list assignment for G_φ, and by the minimality of G, G_φ violates (O). By (14), G_φ contains O_{M1} (the other cases are easily excluded: O_{N2} and O_{N3} since no internal vertex gets a reduced list and O_{P3} since $\ell(P) = 2$). But then w' is adjacent to p_0, and the 2-chord $p_0w'v_{m+2}$ contradicts (12). Therefore, no such coloring φ exists. It follows that $|S_1| = |S_2| = 3$ and $S_1 \subseteq L(v_{m+1})$. Since $L(v_{m+1}) = L(v_{m+2})$, we also have that $S_1 = S_2$. Since $|S_1| = |S_2| = 3$, claim (3)(f) implies that $m = 2$ and $s = 6$. Similarly, we conclude that $L(v_1) = L(p_0) \cup L(v_2)$ and $L(v_6) = L(p_2) \cup L(v_5)$, as otherwise we can color and remove v_1 or v_6.

Let us now consider the case that v_2, v_3 and w' have no common neighbor. If v_1, v_2 and v_3 have no common neighbor, then let φ be an L-coloring of v_2, v_3 and v_4 such that $\varphi(v_4) \notin L(v_5)$. Otherwise, let φ be an L-coloring of v_1, v_3 and v_4 such that $\varphi(v_4) \notin L(v_5)$ and $\varphi(v_1) = \varphi(v_3)$. In the former case, let $G' = G_\varphi$, in the latter case let $G' = G_\varphi - v_2$. Observe that L_φ is a valid list assignment for G' (after possibly adding the edge ww' into M) and that any L_φ-coloring of G' extends to G. Furthermore, G' satisfies (O) by (14), since w' cannot be adjacent to p_0. Hence, G_φ contradicts the minimality of G. Therefore, v_2, v_3 and w' have a common neighbor x', and by symmetry, v_4, v_5 and w have a common neighbor x (see Figure 10).

By (12), we have $x \neq x'$ and x is adjacent neither to p_0 nor to p_2. Furthermore, if $xp_1 \in E(G)$, then consider the cycle $K = p_1p_2v_6v_5x$. Since v_6 has degree at least four, we conclude by (6) that K has two chords incident with v_6. However, that contradicts (7). Therefore, x (and symmetrically x') has no neighbor in P. By (12), neither w nor w' is adjacent to p_0 or p_2. Claims (4) and (5) imply that $x'w, xw', xx' \notin E(G)$. Since both w and w' have degree at least 5, we conclude that each of them has a neighbor that is different from all vertices shown in Figure 10. Suppose that $w'p_1 \notin E(G)$. Then let φ be an L-coloring of x and w' such that $\varphi(x), \varphi(w') \notin L(v_4)$ (note that these colors do not belong to the lists of v_2, v_3 and v_5, as well

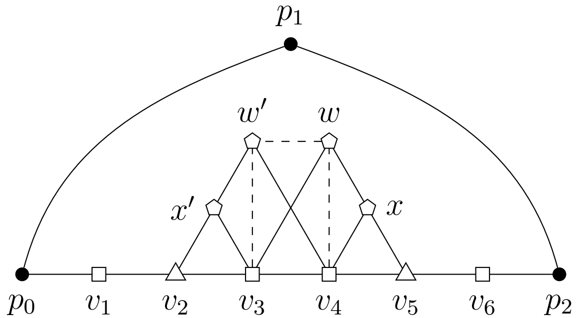


Figure 10: A subcase in the proof when X and X' satisfy (A4) or (A5). The dotted edges may or may not be present.

as to $L(v_1) \setminus L(p_0)$ and $L(v_6) \setminus L(p_2)$). Let $G' = G - \{x, w', v_3, v_4\}$ if $\deg(w) > 5$ and $G' = G - \{x, w', v_3, v_4, w\}$ if $\deg(w) = 5$. Note that G' is not L_φ-colorable since any L_φ-coloring of G' extends to G. Furthermore, the only possible vertices with list of size three in G' are v_2, v_5, w and a common neighbor u of x and w' distinct from w and v_4, if such a vertex exists. By (5), if u exists, then $\deg(w) = 5$ and $w \notin V(G')$. Furthermore, by (5), u and w are not adjacent to v_2 and v_5. Therefore, L_φ is a valid list assignment, the distance condition implies that G' satisfies (O), and thus G' contradicts the minimality of G.

We conclude that $w'p_1 \in E(G)$. Let G_1 and G_2 be the $p_1w'v_4$-components of G, where G_1 contains p_0. Consider an L-coloring of G_2. Note that v_3 has only two neighbors in $G_2 - w'$, thus the coloring of G_2 can be extended to v_3 in such a way that its color is different from the color of w'. Then $G_1 - v_4 + w'v_3$ (with the precolored path $p_0p_1w'v_3$) violates (O). Observe that only v_1 and v_2 have list of size at most four and that x' is a common neighbor of v_3 and w'. Therefore, x' is a vertex in the corresponding obstruction K, and v_2 is a vertex in K with list of size 3. It follows that K is equal to O_{P4}.

However, then $v_1p_1 \in E(G)$, contradicting (7). $\square$

Therefore, $|V(G_q) \cap V(F)| = 1$, and thus each of X and X' satisfies (A5) or (A6). Since $s \geq 3$, we can assume that X' satisfies (A5). Suppose first that X satisfies (A6), and thus $b = 2$. Since $v_1 \notin X$, the inspection of possible cases for X and X' shows that we have $|L(v_2)| = 3$, $X' = \{v_2\}$, and $s = 3$. If v_1, v_2 and v_3 have no common neighbor, then consider any L-coloring φ of v_1 and v_2 such that $\varphi(v_1) \notin L(p_0)$, and observe that G_φ with the list assignment L_φ is a counterexample contradicting the minimality of G: since v_1, v_2, v_3 do not have a common neighbor, we do not get adjacent vertices with lists of size 3; but we may need to add the neighbor z of v_1 along the crossed edge into the set N. The resulting graph satisfies (O), since z is not adjacent to p_0 and p_2 by (12) and (8). Hence, we can assume that v_1, v_2 and v_3 have a common neighbor w, and thus $\deg(v_2) = 3$. Similarly, we conclude that $L(v_1) = L(p_0) \cup L(v_2)$ (if not, we color v_1 with a color in $L(v_1) \setminus (L(p_0) \cup L(v_2))$ and then consider $G' = G - \{v_1, v_2\}$) and that $L(v_3) = L(p_2) \cup L(v_2)$ (if not, we can color v_3 by a color in $L(v_3) \setminus (L(p_2) \cup L(v_2))$ and then consider $G' = G - \{v_2, v_3\}$). By (5), (8) and (12), w has no neighbor in P. Let u be the vertex adjacent to w by the crossed edge, let φ be an L-coloring of w such that $\varphi(w) \notin L(v_2)$ and let $G' = G - \{v_2, w\}$ Note that L_φ is a $(P, N \cup \{u\}, \emptyset)$-valid list assignment for G' and that G' satisfies (O), since no vertex has list of size three. Thus, G' is a counterexample to Theorem 7 contradicting the minimality of G.

Therefore, both X and X' satisfy (A5) and $b = m + 2$. Moreover, (5) implies that the neighbor w' of v_b in $V(G_q) \setminus \{v_{m+1}\}$ is different from w (the neighbor of v_m). Let y be the vertex joined to v_{m+1} by a crossed edge. If $|L(v_{m+1})| \neq 3$, then both X and X' are chosen by cases (X1) or (X3) and $|L(v_m)| = |L(v_{m+2})| = 3$. The condition (A5) implies $|L(v_{m+1})| = 4$. However, in that case we have $|L(v_{m+2})| \neq 3$ both in (X1) and (X3), which is a contradiction. Therefore, $|L(v_{m+1})| = 3$. Consequently, X and X' were chosen by (X2) or (X4) and we have $|L(v_m)|, |L(v_{m+2})| \geq 4$ and $|L(v_{m-1})| = |L(v_{m+3})| = 3$. Since $\deg(v_m) \geq 4$, (12) implies that w

has no neighbor in F other than p_1, v_m and v_{m+1}, and by symmetry, the only possible neighbors of w' in F are p_1, v_{m+1} and v_{m+2}.

Let $S_1 = L(v_{m-1})$ if $m = 3$ and $S_1 = L(v_{m-1}) \setminus L(p_0)$ if $m = 2$. Let $S_2 = L(v_{b+1})$ if $b = s - 2$ and $S_2 = L(v_{b+1}) \setminus L(p_2)$ if $b = s - 1$. By symmetry, we can assume that if $m = 2$, then $b = s - 1$. Let S be the set of colors $c \in L(v_{m+1})$ such that either

(a) $L(v_{m+2}) = S_2 \cup \{c\}$, or

(b) $|L(v_m)| = 4$, $c \notin S_1$ and $S_1 \cup \{c\} \subseteq L(v_m)$.

If $m = 2$, then we have $b = s - 1$, $|S_1| = |S_2| = 2$, there are at most two colors with the property (b) and no colors with the property (a). If $m = 3$, then $|S_1| = 3$ and $|S_2| \leq 3$, there is at most one color with the property (b) and at most one color with the property (a). It follows that $|S| \leq 2$. Let φ be an L-coloring of v_{m-1}, v_{m+1} and v_{m+2} chosen so that $\varphi(v_{m+2}) \notin S_2$, $\varphi(v_{m+1}) \notin S$, $\varphi(v_{m-1}) \in S_1$ and $|L(v_m) \setminus \{\varphi(v_{m-1}), \varphi(v_{m+1})\}| \geq 3$. Note that the choices for $\varphi(v_{m+2})$ and $\varphi(v_{m-1})$ are possible, since $\varphi(v_{m+1})$ does not satisfy (a) and (b), respectively. Consider $G' = G - \{v_{m-1}, v_{m+1}, v_{m+2}\}$ with the list assignment L_φ. By (12), v_{m-1} has no common neighbor with v_{m+1} and none with v_{m+2}, and the only common neighbor of v_{m+1} and v_{m+2} is w'. Therefore, the only vertices with list of size three are v_1 if $m = 3$, v_m, v_{m+3} and w'. Since w' is not adjacent to v_{m+3}, L_φ is $(P, N \cup \{y\}, \emptyset)$-valid. Furthermore, y is adjacent neither to p_0 nor to p_2 by (12), hence G' satisfies (O) by (14) and contradicts the minimality of G. This completes the proof of Theorem 7. □

Proof of Theorem 4. Let G be a graph with crossing number at most two. We may assume that G is nonplanar. Consider a drawing of G in the plane with one or two crossings and let L be a list assignment such that each vertex has five admissible colors. Let xy and uv be two edges crossing each other at the crossing q. Suppose first that the edges xy and uv do not participate in another crossing. Now remove the two edges and add the edges xu, uy, yv, and vx (if they are not already present). This gives rise to a graph G' with at most one crossing, and we can redraw it so that the cycle $xuyv$ bounds the outer face. Now we φ-precolor the path xuy such that $\varphi(x) \neq \varphi(y)$, and give v the list $L(v) \setminus \{\varphi(u)\}$. Theorem 7 now implies that G' has a list coloring which in turn shows that G is L-colorable.

If the edge uv participates in another crossing, then xy does not participate in another one. Suppose that the segment of uv from u to the crossing q does not contain the other crossing. Then we proceed similarly as above: we remove the edges xy and uv and add edges xu and uy. The resulting graph is planar and the path $P = xuy$ is part of a facial walk. Thus we may φ-precolor the path so that $\varphi(x) \neq \varphi(y)$ and then remove $\varphi(u)$ from the list of v. Now, we apply Theorem 7 with $N = \{v\}$ to obtain a coloring that again confirms that G is L-colorable. □

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